A new image reconstruction technique using the Linear Sampling Method in inverse-scattering problems

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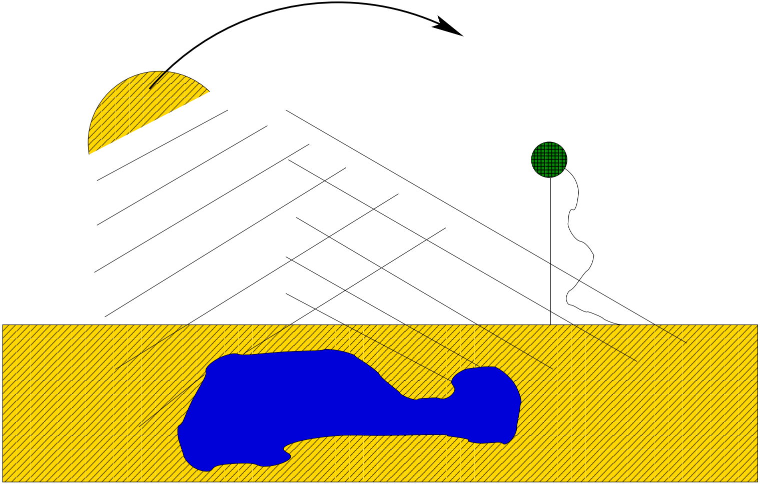
The problem (1)

The name of the game:

Discover the shape of hidden objects

- objects typically hidden in another medium (ground, water, body, . . .)
- illuminate the objects by plane waves from “”all” directions
- measure waves scattered (reflected) by the object in “all” directions
- reconstruct the object

The problem (2)



The Linear Sampling Method (1)

The direct problem:

- the hidden object $\mathcal{O}$ is given
- the incident plane wave

$$u^{inc}(x,d) = e^{ik\langle x,d\rangle} \quad d \in \mathcal{S}$$

(k is the wave number)

- the PDE model: find $u^s(x,d)$ (the scattered wave) such that

$$\Delta u^s + k^2 u^s = 0 \quad \text{(outside } \mathcal{O}\text{)}$$

with

$$u = u^{ins} + u^s, \qquad u = 0 \;\text{ on }\; \partial\mathcal{O}$$

and

$$\lim_{r\to\infty} \frac{\partial u^s}{\partial r} - iku^s = 0.$$

The Linear Sampling Method (2)

The inverse problem problem:

- the hidden object $\mathcal{O}$ is given
- one knows the far-field u^∞ such that

$$u(x,d) = \left[\frac{e^{ik\langle x,d\rangle}}{x} u^\infty \left(\frac{x}{\|x\|}, d \right) + O\left(\frac{1}{x}\right) \right] \quad \text{when} \quad \|x\| \to \infty$$

- assuming the same PDE model, find $\delta\mathcal{O}$.

A 0-1 formulation:

Given u^∞, decide, for each $x \in \mathbb{R}^3$, whether $x \in \mathcal{O}$ or not.

The Linear Sampling Method (3)

For $\hat{x} = x/\|x\|$, define the far-field operator and equation

$$(\mathcal{F}g)(\hat{x}) \stackrel{\text{def}}{=} \int_{\mathcal{S}} u^{\infty}(\hat{x}, d) g(d)\, ds(d) = \frac{1}{4\pi} e^{-ik\langle \hat{x}, d\rangle}$$

(the far-field pattern associated to the plane wave $e^{ik\langle \hat{x}, d\rangle}$)

Finding g is ill-posed!

But (Colton) ...

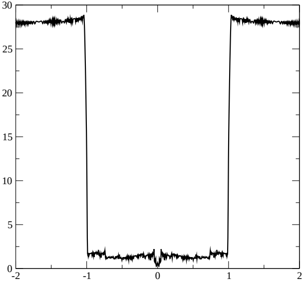
$$\forall \epsilon \quad \exists g_{\epsilon} \quad \|\mathcal{F}g_{\epsilon}(\cdot, d) - \tfrac{1}{4\pi} e^{-ik\langle \cdot, d\rangle}\| \leq \epsilon$$

The Linear Sampling Method (4)

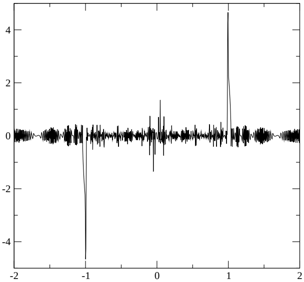
The crucial property:

$$\lim_{z \in \mathcal{O}, z \to \partial\mathcal{O}} \|g_\epsilon(\cdot, z)\| = \infty, \quad \text{and} \quad \|g_\epsilon(\cdot, z)\| = \infty \quad \text{for} \quad (z \in \mathbb{R}^3 \setminus \mathcal{O})$$

For $\mathcal{O}$ a sphere: compute the shape of $\mathcal{O}$ from the level curves of $\|g_\epsilon\|$



$\log_{10} \|g_\epsilon\|$



$\log_{10} \|\nabla g_\epsilon\|$

The Linear Sampling Method (5)

Approximate the far-field equation

$$\int_{\mathcal{S}} u^{\infty}(\hat{x}, d) g(d)\, ds(d) = \frac{1}{4\pi} e^{-ik\langle \hat{x}, d\rangle}$$

by its discretized form

$$\sum_{j=1}^{N} F_{\ell,j} g_j(z) = e^{-ik\langle d_\ell, z\rangle}$$

i.e.

$$\boxed{Fg(z) = b^{\infty}(z) \quad (z \in \mathcal{B} \subset \mathbb{R}^3)}$$

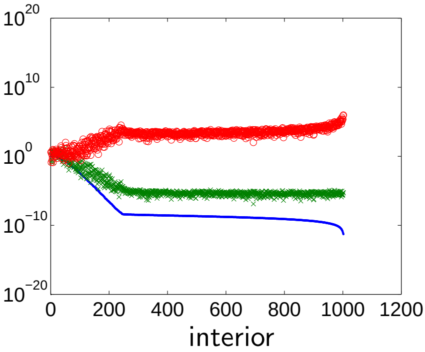
This is an underdetemined system! $\Rightarrow$ regularization (ignore nullspace)

Regularizing the discretized far-field equation

Assume $F = U\Sigma V^*$. Well-known regularization techniques (Tikhonov-Morosov, TSVD, L-curve, GCV, ...) applicable if the Picard coefficients

$$\left\{ \frac{u_\ell^* b^\infty(z)}{\sigma_\ell} \right\} \quad \text{decrease to zero.}$$

However,



Operator perturbation and clustered eigenspace recovery

Use a different approach! Consider

$$A = Q^* \Lambda Q \quad \text{and} \quad \bar{A} = A + tE = \bar{Q}^* \bar{\Lambda} \bar{Q}$$

Choose $b = \sum_{j=1}^{n} \alpha_j q_j$. Then, after some analysis and for t small,

$$\bar{q}_\ell^* b \approx \alpha_\ell + t \sum_{j \neq \ell} \alpha_j q_\ell^* E^* P_\ell (\lambda_\ell I_{n-1} - \Lambda_\ell)^{-1} e_{j\ell}$$

where $P_\ell = Q$ minus its ℓ-th column.

$|\bar{q}_\ell^* b|$ large whenever t is small, ℓ is the index of a clustered eigenvalue and b has a significant component along the cluster's eigenvectors

Note: A, Q or Λ may be unknown!

Application to Linear Sampling

Choose

$$\bar{A} = FF^*, \quad \text{and} \quad tE = FF^* - F^\infty (F^\infty)^*$$

and thus

$$\bar{Q} = U \quad \text{and} \quad |\bar{q}_\ell^* b| = u_\ell^* b^\infty.$$

$|u_\ell^* b^\infty|$ large whenever ℓ is large, $\|FF^* - F^\infty (F^\infty)^*\|$ is small and b^∞ has a significant component along the (unknown) approximate nullspace of the far-field operator

(as observed)

reconstruct nullspace!

The SVD-tail algorithm

1. Select $\mathcal{T}$ a d-dimensional subspace (approximately) spanned by the the d leftmost singular vectors of F
2. Compute $\{w_\ell\}$ a basis of $\mathcal{T}$ and $\vartheta_\ell(z) = w_\ell^* b^\infty(z) \quad (\ell = 1, \ldots, d)$
3. Define

$$\psi_d(z) = \frac{1}{\|\vartheta(z)\|}$$

$\psi_d(z)$ small for z outside $\mathcal{O}$ and large for z inside $\mathcal{O}$

Same as $g_\epsilon(z)$ but MUCH cheaper to evaluate (no full SVD)!

Use level curves of ψ_d to compute $\partial\mathcal{O}$!

(isovalue heuristic)

Illustration: the unknown objects

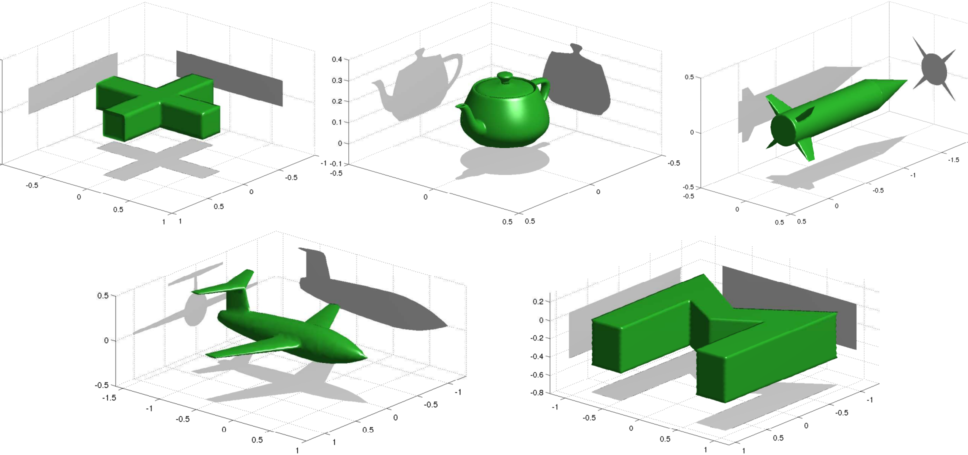
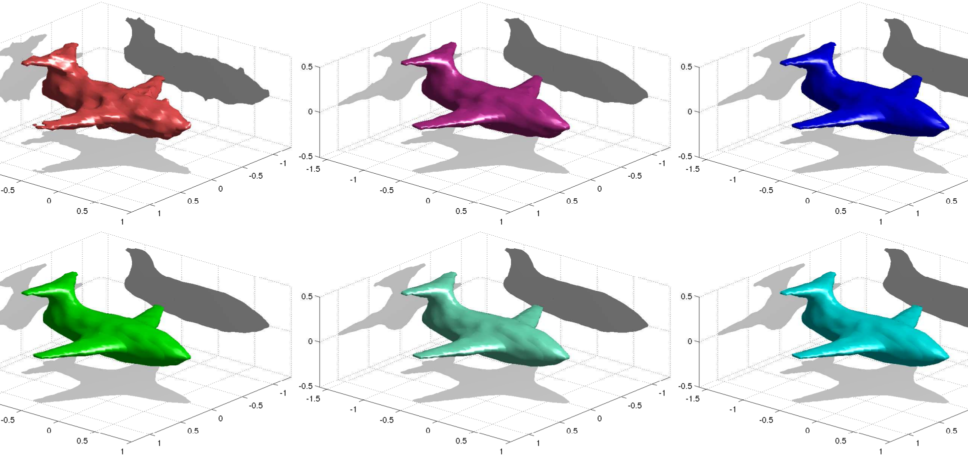


Illustration: the reconstruction of the plane

$N = 1002, d = 5, 20, 35, 50, 65, 80$



Conclusions

- Interesting (to me) 0-1 decision problem in 3D-space
- Efficient computaional scheme
- Uses level curves of a nullspace "indicator"
- Can this be extended to other such problems ?

Thank you for your attention!