Nonlinear programming without a penalty function or filter

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Outline

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1. The new method
2. Convergence theory
3. Numerical experiments
4. Conclusions

Outline

The problem

Consider the equality constrained problem

$$\min_{x \in \mathbb{R}^n} \quad f(x)$$
$$\text{such that} \quad c(x) = 0.$$

Motivation:

- large-scale problems
- PDE constrained optimal control applications
- backup for a new filter method (in development)
- personnal curiosity...

The pedigree and context

An inexact SQP algorithm

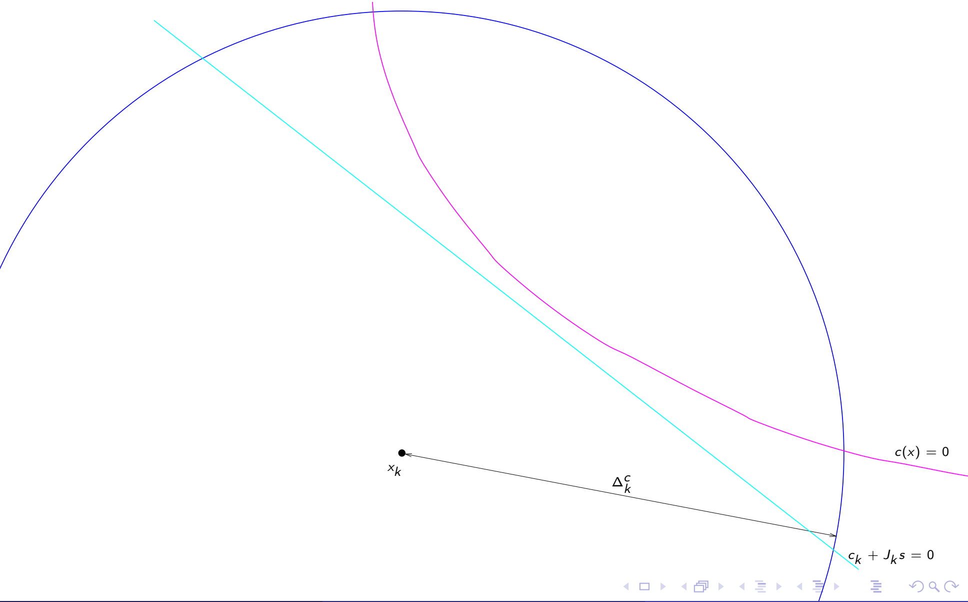
Origin/ inspiration from

- Himmelblau (1971) "flexible tolerance"
- Byrd-Omojokun (1989) SQP technique ...
- ... and much of the SQP literature (e.g. Nocedal-Plantenga, 1998)

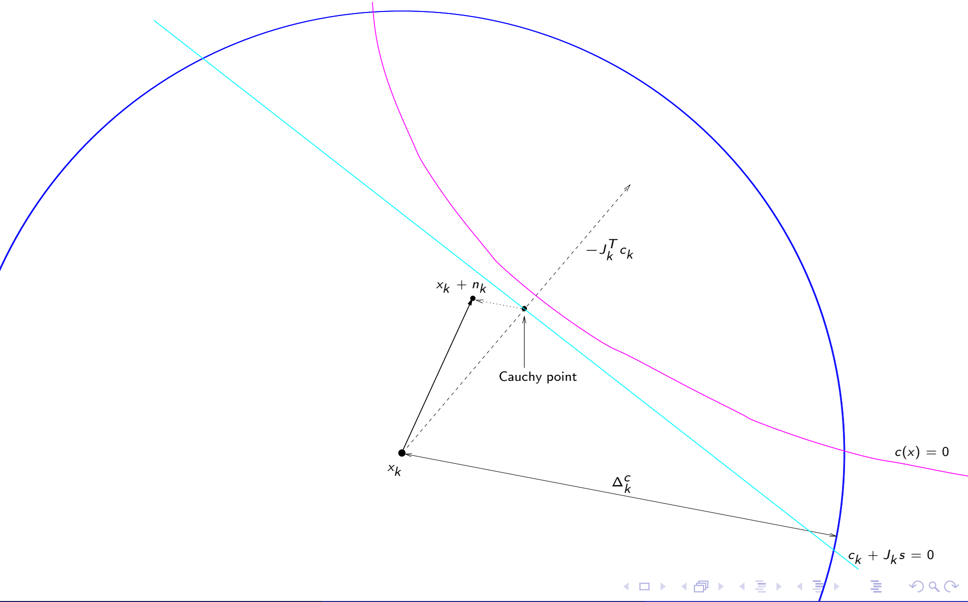
Similar concerns addressed by

- Byrd-Curtis-Nocedal (2006)
- (a posteriori) Zoppke-Donaldson (1995) "tolerance tube" method

At a typical iterate. . .



The normal step (pic)



The normal step (equs)

Constraint violation:

$$\theta(x) = \tfrac{1}{2}\|c(x)\|^2$$

Compute a Gauss-Newton step n_k for the problem $\min_n \theta(x_k + n)$ inside the constraints' trust region:

$$n_k \in \{n \in \mathbf{R}^n \mid \|n\| \le \Delta_k^c\}$$

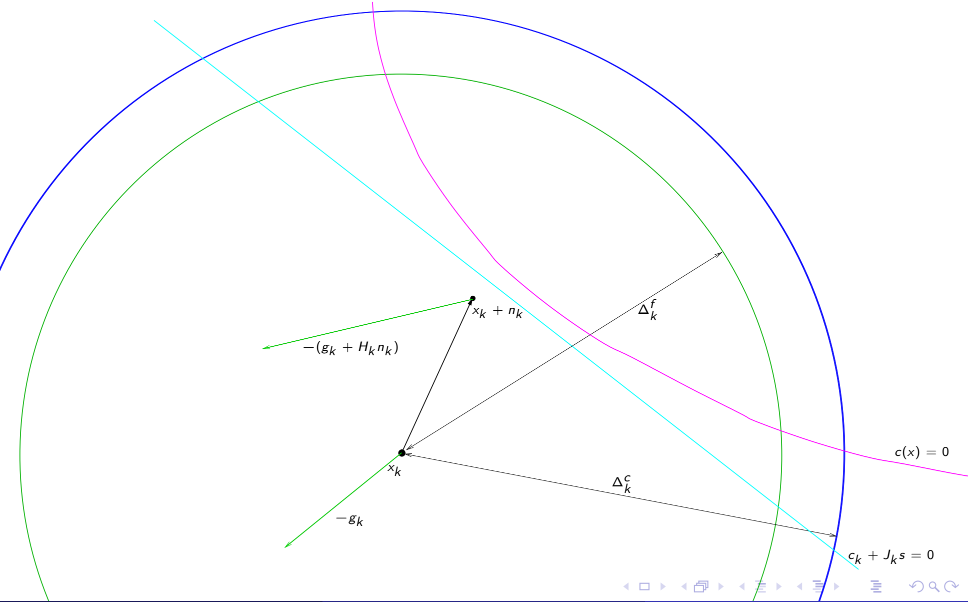
and also

$$\|n_k\| \le \kappa_{\rm n} \|c_k\|.$$

Relevant Cauchy condition:

$$\tfrac{1}{2}\|c_k\|^2 - \tfrac{1}{2}\|c_k + J_k n_k\|^2 \ge \kappa_{\rm nC} \|J_k^T c_k\| \min\left[\frac{\|J_k^T c_k\|}{1 + \|J_k^T J_k\|}, \Delta_k^c\right]$$

Objective's model and gradients (pic)



Objective's model and gradients (equs)

Whenever

$$\|n_k\| \leq \kappa_{\rm B} \Delta_k$$

decrease the objective's model:

$$m_k(x_k + n_k + t) = f_k + \langle g_k + H_k n_k, t\rangle + \tfrac{1}{2}\langle t, H_k t\rangle$$

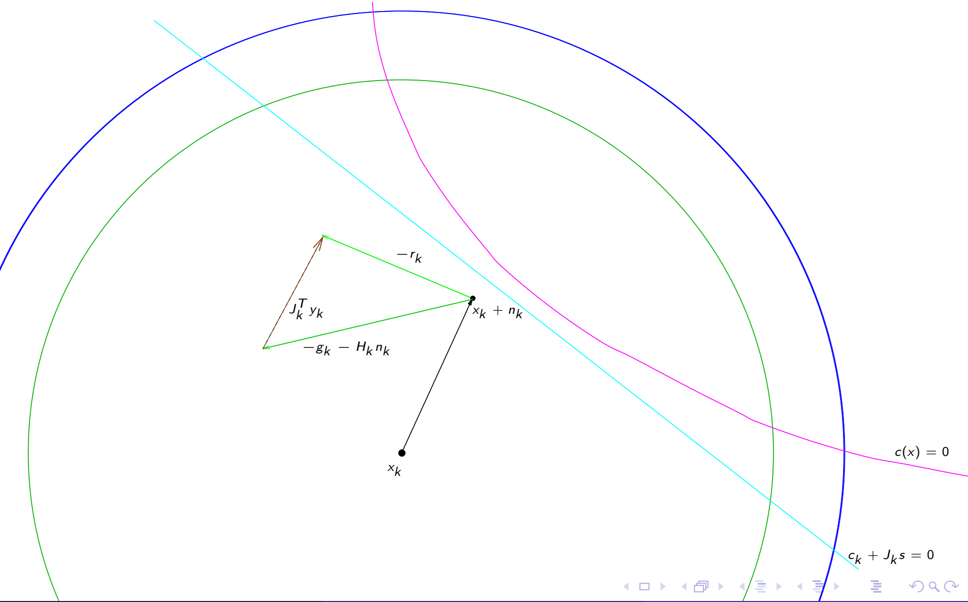
The objective's trust region

$$\{t \in \mathbf{R}^n \mid \|t\| \leq \Delta_k^f\}$$

The composite trust region:

$$s_k \in \mathcal{B}_k = \mathcal{N}_k \cap \mathcal{T}_k = \{s \in \mathbf{R}^n \mid \|s\| \leq \Delta_k = \min[\Delta_k^c, \Delta_k^f]\},$$

The effect of Lagrange multipliers (pic)



The effect of Lagrange multipliers (equs)

When sufficiently inside, i.e.,

$$\|n_k\| \leq \kappa_{\rm B} \Delta_k$$

compute least-squares Lagrange multipliers estimates from

$$\|y_k + [J_k^T]^I (g_k + H_k n_k)\| \leq \omega_1(\|c_k\|)$$

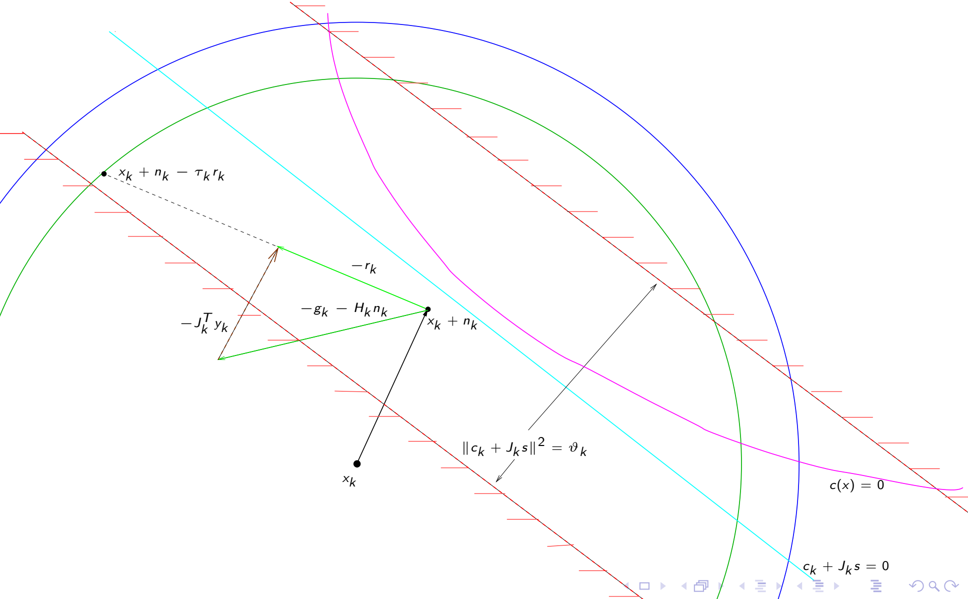
such that the approximate projected gradient

$$r_k \stackrel{\rm def}{=} g_k + H_k n_k + J_k^T y_k$$

satisfies

$$\langle g_k + H_k n_k, r_k \rangle \geq 0 \qquad \text{and} \qquad \|r_k\| \leq \kappa_{\rm nr} \|g_k + H_k n_k\|$$

Limiting infeasibility (pic)



Limiting infeasibility (equs)

Avoid jeopardizing the gain in feasibility due to the normal step

$$\|c_k + J_k(n_k + t_k)\|^2 \le \kappa_{\rm nt}\|c_k\|^2 + (1-\kappa_{\rm nt})\|c_k + J_k n_k\|^2 = \vartheta_k$$

for $\kappa_{\rm nt} \in (0,1)$, but also

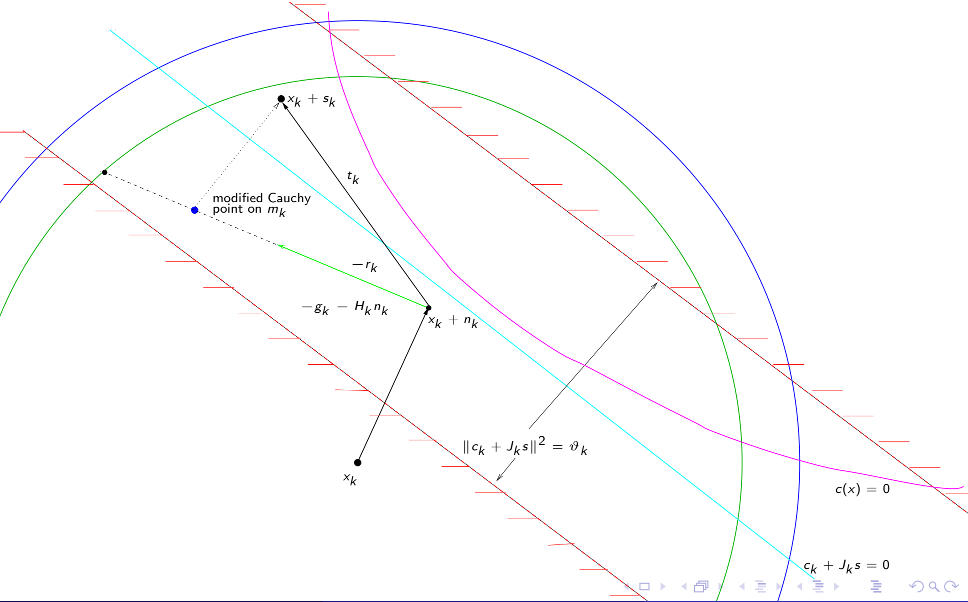
$$\|c_k + J_k(n_k - \tau_k r_k)\|^2 \le \vartheta_k$$

where

$$\tau_k = \frac{-\beta_k + \sqrt{\beta_k^2 + \Delta_k^2 - \|n_k\|^2}}{\|r_k\|} \quad \text{with} \quad \beta_k \stackrel{\rm def}{=} \frac{\langle n_k, r_k\rangle}{\|r_k\|}$$

is the maximum stepsize such that $x_k + n_k - \tau r_k$ remains in the composite trust region

The complete step (pic)



The complete step (equs)

(Maybe) compute a tangential step t_k that reduces m_k beyond the modified Cauchy point:

$$m_k(x_k+n_k)-m_k(x_k+n_k+t_k) \geq \kappa_{\rm tC1}\pi_k \min\left[\frac{\pi_k}{1+\|H_k\|}, \tau_k\|r_k\|\right]$$

where

$$\pi_k \stackrel{\rm def}{=} \frac{\langle g_k + H_k n_k, r_k\rangle}{\|r_k\|}$$

(also used as dual feasibility measure)
Define

$$s_k = n_k + t_k \quad \text{and} \quad \delta_k^c = \tfrac{1}{2}\|c_k\|^2 - \tfrac{1}{2}\|c_k + J_k s_k\|^2$$

f-steps and c-steps

Say that s_k is a f-step whenever

- $$\|n_k\| \leq \kappa_{\rm B}\Delta_k \quad \text{and} \quad \pi_k > \omega_2(\|c_k\|) \qquad (t_k \text{ computed})$$
- $$\delta_k^f \stackrel{\rm def}{=} m_k(x_k) - m_k(x_k + s_k) \geq \kappa_\delta[m_k(x_k + n_k) - m_k(x_k + s_k)]$$
- $$\theta(x_k^+) \leq \theta_k^{\max} \quad \text{for some} \quad \theta_k^{\max} > \theta(x_k)$$

($\theta_k^{\max}$ is the maximal infeasibity tolerated at iteration k)

Otherwise, s_k is a c-step

Accepting f-steps

Accept a f-step if

$$\rho_k^f \stackrel{\text{def}}{=} \frac{f(x_k) - f(x_k^+)}{\delta_k^f} \geq \eta_1$$

and update the f-radius by

$$\Delta_{k+1}^f \in \begin{cases} [\Delta_k^f, \infty) & \text{if } \rho_k^f \geq \eta_2, \\ [\gamma_2 \Delta_k^f, \Delta_k^f] & \text{if } \rho_k^f \in [\eta_1, \eta_2), \\ [\gamma_1 \Delta_k^f, \gamma_2 \Delta_k^f] & \text{if } \rho_k^f < \eta_1, \end{cases}$$

Also possibly increase the c-radius by

$$\Delta_{k+1}^c \in [\Delta_k^c, +\infty) \quad \text{if} \quad \theta(x_k^+) \leq \eta_3 \theta_k^{\max} \quad \text{and} \quad \rho_k^f \geq \eta_1$$

Accepting c-steps

Accept a f-step if

$$\rho_k^c \stackrel{\text{def}}{=} \frac{\theta(x_k) - \theta(x_k^+)}{\delta_k^c} \geq \eta_1$$

and update the c-radius by

$$\Delta_{k+1}^c \in \begin{cases} [\Delta_k^c, \infty) & \text{if } \rho_k^c \geq \eta_2, \\ [\gamma_2 \Delta_k^c, \Delta_k^c] & \text{if } \rho_k^c \in [\eta_1, \eta_2), \\ [\gamma_1 \Delta_k^c, \gamma_2 \Delta_k^c] & \text{if } \rho_k^c < \eta_1 \end{cases}$$

Also decrease the maximal infeasibility whenever $\rho_k^c \geq \eta_1$ by

$$\theta_{k+1}^{\max} = \max\left[\kappa_{\text{tx1}} \theta_k^{\max}, \theta(x_k^+) + \kappa_{\text{tx2}}(\theta(x_k) - \theta(x_k^+))\right]$$

Global convergence to first-order points

Assumptions:

- f and c are smooth and have bounded derivatives at x_k
- f is bounded below on $\{x \in \mathbb{R}^n \mid \theta(x) \le \theta_0^{\max}\}$
- the Jacobian is full-rank on the feasible set

Main result:

Either there exists a subsequence indexed by $\mathcal{Z}$ such that

$$\lim_{k\to\infty, k\in\mathcal{Z}} \|J_k^T c_k\| = 0 \quad \text{with} \quad \liminf_{k\to\infty, k\in\mathcal{Z}} \|c_k\| > 0,$$

or

$$\lim_{k\to\infty} \|c_k\| = 0 \quad \text{and} \quad \liminf_{k\to\infty} \|g_k + J_k^T y_k\| = 0.$$

Sketch of the proof: technical lemmas

- $$\theta(x_j) < \theta_k^{\max} \quad (j \geq k)$$

- $$\delta_k^c \geq \kappa_{\rm nC2} \|J_k^T c_k\| \min\left[\frac{\|J_k^T c_k\|}{1+\|J_k^T J_k\|}, \Delta_k^c\right]$$

- $$\text{first-order criticality} \quad \Leftrightarrow \|c_k\| \rightarrow 0 \text{ and } \pi_k \rightarrow 0$$

- $$\pi_k \geq \epsilon_f \text{ for all } f\text{-iterations } \Rightarrow \Delta_k^f \geq \epsilon_{\mathcal{F}}$$
  $$\|J_k^T c_k\| \geq \epsilon_\theta \text{ for all } c\text{-iterations} \Rightarrow \Delta_k^c \geq \epsilon_{\mathcal{C}}$$

- $$\|c_k\| \rightarrow 0 \quad \Rightarrow \quad \text{very successful } c\text{-iterations}$$

Sketch of the proof: the works

- f-iterations only $\Rightarrow \|c_k\| \to 0$ and $\pi_k \to 0$

(f monotone + modified Cauchy condition + infeasibility dominated)

- c-iterations $\Rightarrow \|c_k\| \to 0$ and $\Delta_k^c \geq \epsilon$

(shrinking $\theta_k^{\max}$!!)

- c-iterations $\Rightarrow \pi_k \to 0$

($\Delta_k \geq \epsilon$ + infinite subsequences of c-iterations impossible)

May include Lagrange multipliers in model for tangential step!

Numerical experiments: context

Results are for the following choices:

- normal step by TCG on $\|c_k + J_k n\|_2$ within $\|n\|_2 \leq \Delta_k^c$
- multipliers by factorization if J_k is full rank or CG otherwise
- tangential step by GLTR without preconditioner
- Attempts are made to identify (and then remove) dependent linearized constraints

Code still very experimental!!!

CUTEr equality constrained problems 1

Name	n	m	iter	ngeval	*f*	*c*	time
ALLINITC	2	1	9	8	-1.00e+00	2.15e-11	0.03
BT1	2	1	15	7	-1.00e-00	1.09e-11	0.05
BT2	3	1	35	20	3.26e-02	1.78e-15	0.11
BT3	5	3	6	6	4.09e+00	0.00e+00	0.03
BT4	3	2	8	7	-4.55e+01	0.00e+00	0.03
BT5	3	2	15	9	9.62e+02	5.33e-15	0.05
BT6	5	2	22	11	2.77e-01	3.26e-13	0.09
BT7	5	3	40	22	3.06e+02	7.02e-12	0.13
BT8	5	2	10	10	1.00e+00	3.81e-06	0.05
BT9	4	2	28	17	-1.00e+00	2.50e-08	0.08
BT10	2	2	7	7	-1.00e+00	4.18e-09	0.21
BT11	5	3	9	9	8.25e-01	1.05e-12	0.12
BT12	5	3	7	6	6.19e+00	1.29e-08	0.03
BYRDSPHR	3	2	13	9	-4.68e+00	8.52e-09	0.03
DIXCHLNG	10	5	31	16	2.47e+03	2.22e-16	0.13
EIGENA2	110	55	4	4	0.00e+00	0.00e+00	0.06
EIGENACO	110	55	4	4	0.00e+00	0.00e+00	0.07
EIGENB2	110	55	3	3	1.80e+01	0.00e+00	0.06
EIGENBCO	110	55	3	3	9.00e+00	0.00e+00	0.07
EIGENC2	462	231	14	9	1.37e-18	1.04e-11	31.81
EIGENCCO	462	231	46	30	1.03e-25	2.22e-16	171.12

CUTEr equality constrained problems 2

Name	n	m	iter	ngeval	f	c	time
ELEC	150	50	287	132	1.06e+03	1.27e-09	4.75
FCCU	19	8	5	5	1.11e+01	3.55e-15	0.09
GRIDNETE	60	36	7	7	3.96e+01	4.44e-16	0.21
GRIDNETH	264	144	7	7	5.71e+01	4.44e-16	1.52
HS6	2	1	13	11	4.34e-24	7.77e-15	0.04
HS7	2	1	10	8	-1.73e+00	2.35e-07	0.04
HS8	2	2	8	6	-1.00e+00	6.53e-12	0.02
HS9	2	1	4	3	-5.00e-01	0.00e+00	0.01
HS26	3	1	24	19	6.36e-11	4.48e-06	0.08
HS27	3	1	17	17	4.00e-02	1.04e-32	0.06
HS28	3	1	3	3	1.23e-32	0.00e+00	0.01
HS39	4	2	28	17	-1.00e+00	2.50e-08	0.07
HS40	4	3	6	6	-2.50e-01	1.11e-16	0.02
HS42	4	2	4	4	1.39e+01	5.21e-09	0.03
HS46	5	2	16	16	9.67e-10	7.57e-06	0.07
HS47	5	3	16	16	5.13e-10	6.35e-07	0.06
HS47	5	3	16	16	5.13e-10	6.35e-07	0.06
HS48	5	2	3	3	0.00e+00	0.00e+00	0.01
HS49	5	2	16	16	6.96e-09	8.88e-16	0.06
HS50	5	3	9	9	4.93e-32	8.88e-16	0.03
HS51	2	1	6	6	0.00e+00	0.00e+00	0.02

CUTEr equality constrained problems 3

Name	n	m	iter	ngeval	f	c	time
HS51	5	3	3	3	0.00e+00	0.00e+00	0.01
HS52	5	3	3	3	5.33e+00	1.39e-17	0.01
HS55SIM	1	1	2	2	0.00e+00	0.00e+00	0.02
HS56MOD	7	4	107	55	-3.46e+00	3.11e-15	0.22
HS56	7	4	107	55	-3.46e+00	3.11e-15	0.21
HS61	3	2	7	6	-1.44e+02	7.20e-11	0.03
HS77	5	2	11	11	2.42e-01	2.52e-14	0.04
HS78	5	3	6	6	-2.92e+00	5.55e-16	0.02
HS79	5	3	6	6	7.88e-02	4.78e-13	0.02
HS100LNP	7	2	15	10	6.81e+02	1.42e-14	0.07
HS111LNP	10	3	17	13	-4.78e+01	4.72e-09	0.08
KOPPEL	12	6	4	3	4.50e+00	2.98e-09	0.03
LCH	150	1	159	63	-4.23e+00	5.98e-12	3.72
LUKVLE1	100	98	13	10	6.23e+00	8.88e-16	0.22
LUKVLE3	100	2	11	11	2.76e+01	1.22e-15	0.15
LUKVLE6	99	49	16	16	6.04e+03	2.86e-16	0.40
LUKVLE7	100	4	15	11	-2.59e+01	1.01e-16	0.21
LUKVLE8	100	98	19	13	1.42e+04	3.71e-16	0.30
LUKVLE9	100	6	67	28	1.12e+01	1.42e-14	0.62
LUKVLE10	100	98	31	17	3.49e+01	6.66e-16	0.50

CUTEr equality constrained problems 4

Name	n	m	iter	ngeval	f	c	time
LUKVLE11	98	64	78	36	4.63e+03	2.66e-14	1.39
LUKVLE13	98	64	26	17	7.90e+02	4.00e-15	0.75
LUKVLE14	98	64	53	29	2.60e+00	2.97e-10	0.77
LUKVLE15	97	72	83	48	4.05e-08	1.11e-15	3.05
LUKVLE16	97	72	23	18	4.72e+01	6.66e-16	0.21
LUKVLE17	97	72	1000	712	3.24e+02	1.65e-18	11.20
LUKVLE18	97	72	1000	659	1.08e+02	1.12e-17	5.96
LUKVLESC	98	64	53	29	2.60e+00	2.97e-10	0.80
MARATOS2	2	1	4	4	-1.00e+00	7.99e-09	0.02
MARATOS	2	1	4	4	-1.00e+00	7.99e-09	0.01
MWRIGHT	5	3	8	8	2.50e+01	1.11e-15	0.03
ORTHRDM2	203	100	12	8	7.78e+00	1.14e-13	0.56
ORTHRDS2	103	50	1000	905	2.40e+01	8.53e-14	3.42
ORTHREGA	133	64	41	24	3.50e+02	4.51e-16	0.55
ORTHREGB	27	6	39	19	9.83e-27	7.51e-13	0.29
ORTHREGC	105	50	15	10	1.98e+00	5.26e-16	0.25
ORTHREGD	103	50	1000	894	7.20e+01	2.84e-13	7.44
ORTHRGDM	23	10	1000	870	2.02e+02	1.14e-13	5.76
ORTHRGDS	155	76	18	13	2.34e+01	8.53e-14	0.27
ROBOT	14	2	11	7	0.00e+00	2.95e-12	0.22
S316-322	2	1	3	3	3.34e+02	5.55e-17	0.01

Conclusions

Encouraging so far!

- finalize the algorithm
- inequality constraints
- theory to be completed
- current code consolidation necessary. . .
- many pending implementation issues
- include a filter mechanism

Thanks for your attention