Second-order convergence properties of trust-region methods using incomplete curvature information, with an application to multigrid optimization

by S. Gratton[1], A. Sartenaer[2] and Ph. L. Toint[2]

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[1] CERFACS,
av. G. Coriolis, Toulouse, France,
Email: serge.gratton@cerfacs.fr

[2] Department of Mathematics,
University of Namur,
61, rue de Bruxelles, B-5000 Namur, Belgium,
Email: annick.sartenaer@fundp.ac.be, philippe.toint@fundp.ac.be

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Serge Gratton, Annick Sartenaer and
Philippe L. Toint

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Abstract

Convergence properties of trust-region methods for unconstrained nonconvex optimization is considered in the case where information on the objective function's local curvature is incomplete, in the sense that it may be restricted to a fixed set of "test directions" and may not be available at every iteration. It is shown that convergence to local "weak" minimizers can still be obtained under some additional but algorithmically realistic conditions. These theoretical results are then applied to recursive multigrid trust-region methods, which suggests a new class of algorithms with guaranteed second-order convergence properties.

Keywords: nonlinear optimization, convergence to local minimizers, multilevel problems.

1 Introduction

It is highly desirable that iterative algorithms for solving nonconvex unconstrained optimization problems of the form

$$\min_{x \in \mathbb{R}^n} f(x) \tag{1.1}$$

converge to solutions that are local minimizers of the objective function f, rather than mere first-order critical points, or, at least, that the objective function f becomes asymptotically convex. Trust-region methods (see Conn, Gould and Toint [2] for an extensive coverage) are well-known for being able to deliver these guarantees under assumptions that are not too restrictive in general. In particular, it may be proved that every limit point of the sequence of iterates satisfies the second-order necessary optimality conditions under the assumption that the smallest eigenvalue of the objective function's Hessian is estimated at each iteration. There exist circumstances, however, where this assumption appears either costly or irrealistic. We are in particular motivated by multigrid recursive trust-region methods of the type investigated in

[3]: in these methods, gradient smoothing is achieved on a given grid by successive coordinate minimization, a procedure that only explores curvature along the vectors of the canonical basis. As a consequence, some negative curvature directions on the current grid may be undetected. Moreover, these smoothing iterations are intertwined with recursive iterations which only give information on coarser grids. As a result, information on negative curvature directions at a given iteration may either be incomplete or simply missing, causing the assumption required for second-order convergence to fail. Another interesting example is that of algorithms in which the computation of the step may require the explicit determination of the objective function Hessian's smallest eigenvalue in order to take negative curvature into account (see, for instance, [5]). Because of cost, one might then wish to avoid the eigenvalue calculation at every iteration, which again jeopardizes the condition ensuring second-order convergence.

Our purpose is therefore to investigate what can be said about second-order convergence of trust-region methods when negative curvature information is incomplete or missing at some iterations. We indicate that a weaker form of second-order optimality may still hold at the cost of imposing a few additional assumptions that are algorithmically realistic. Section 2 introduces the necessary modifications of the basic trust-region algorithm, whose second-order convergence properties are then investigated in Section 3. Application to the recursive multigrid trust-region methods is then discussed in more detail in Section 4. Some conclusions and extensions are finally proposed in Section 5.

2 A trust-region algorithm with incomplete curvature information

We consider the unconstrained optimization problem (1.1), where f is a twice-continuously differentiable objective function which maps $\mathbb{R}^n$ into $\mathbb{R}$ and is bounded below. We are interested in using a trust-region algorithm for solving (1.1). Methods of this type are iterative and, given an initial point x_0, produce a sequence $\{x_k\}$ of iterates (hopefully) converging to a local minimizer of the problem, i.e., to a point x_* where $g(x_*) \stackrel{\text{def}}{=} \nabla_x f(x_*) = 0$ (first-order convergence) and $\nabla_{xx} f(x_*)$ is positive semi-definite (second-order convergence). At each iterate x_k, classical trust-region methods build a model $m_k(x_k + s)$ of $f(x_k + s)$. This model is then assumed to be adequate in a "trust region" $\mathcal{B}_k$, defined as a sphere of radius $\Delta_k > 0$ centered at x_k, i.e.,

$$\mathcal{B}_k = \{x_k + s \in \mathbb{R}^n \mid \|s\| \leq \Delta_k\},$$

where $\|\cdot\|$ is the Euclidean norm. A step s_k is then computed that "sufficiently reduces" this model in this region, which is typically achieved by (approximately) solving the subproblem

$$\min_{\|s\| \leq \Delta_k} m_k(x_k + s).$$

The objective function is then computed at the trial point $x_k + s_k$ and this trial point is accepted as the next iterate if and only if the ratio

$$\rho_k \stackrel{\text{def}}{=} \frac{f(x_k) - f(x_k + s_k)}{m_k(x_k) - m_k(x_k + s_k)} \tag{2.1}$$

is larger than a small positive constant η_1. The value of the radius is finally updated to ensure that it is decreased when the trial point cannot be accepted as the next iterate, and is increased or unchanged if ρ_k is sufficiently large. In many practical trust-region algorithms, the model $m_k(x_k + s)$ is quadratic and takes the form

$$m_k(x_k + s) = f(x_k) + \langle g_k, s\rangle + \tfrac{1}{2}\langle s, H_k s\rangle, \tag{2.2}$$

where

$$g_k \stackrel{\text{def}}{=} \nabla_x m_k(x_k) = \nabla_x f(x_k), \tag{2.3}$$

H_k is a symmetric $n \times n$ approximation of $\nabla_{xx} f(x_k)$ and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. If the model is not quadratic, it is assumed that it is twice-continuously differentiable and that the conditions $f(x_k) = m_k(x_k)$ and (2.3) hold. The symbol H_k then denotes the model's Hessian at the current iterate, that is,

$$H_k = \nabla_{xx} m_k(x_k).$$

We refer the interested reader to [2] for an extensive description and analysis of trust-region methods.

A key for the convergence analysis is to clarify what is precisely meant by "sufficient model reduction". It is well-known that, if the step s_k satisfies the *Cauchy condition*

$$m_k(x_k) - m_k(x_k + s_k) \geq \kappa_{\text{red}} \|g_k\| \min\left[\frac{\|g_k\|}{1 + \|H_k\|}, \Delta_k\right] \tag{2.4}$$

for some constant $\kappa_{\text{red}} \in (0, 1)$, then one can prove first-order convergence, i.e.

$$\lim_{k \to \infty} \|g_k\| = 0, \tag{2.5}$$

under some very reasonable assumptions (see [2, Theorem 6.4.6]). Thus limit points x_* of the sequence of iterates are *first-order critical*, which is to say that $\nabla_x f(x_*) = 0$. Interestingly, obtaining a step s_k satisfying the Cauchy condition does not require the knowledge of the complete Hessian H_k, as could possibly be inferred from the term $\|H_k\|$ in (2.4). In fact, minimization of the model in the intersection of the steepest descent direction with $\mathcal{B}_k$ is sufficient, which only requires the knowledge of $\langle g_k, H_k g_k\rangle$, that is model curvature along a single direction.

Under some additional assumptions, it is also possible to prove that every limit point x_* of the sequence of iterates is *second-order critical* in that $\nabla_{xx} f(x_*)$ is positive semi-definite. In particular, and most crucially for our present purpose, one has to assume that, whenever τ_k, the smallest eigenvalue of H_k, is negative, the model exploits this information in the sense that

$$m_k(x_k) - m_k(x_k + s_k) \geq \kappa_{\text{sod}} |\tau_k| \min[\tau_k^2, \Delta_k^2] \tag{2.6}$$

for some constant $\kappa_{\text{sod}} \in (0, \frac{1}{2})$ [2, Assumption AA.2, p. 153].

As indicated above, our objective is to weaken (2.6) in two different ways. First, we no longer assume that the smallest eigenvalue of H_k is available, but merely that we can compute

$$\chi_k = \min_{d \in \mathcal{D}} \langle d, H_k d \rangle, \tag{2.7}$$

where $\mathcal{D}$ is a fixed (finite or infinite) closed set of normalized *test directions* d. Moreover, the value of χ_k will only be known at a subset of iterations indexed by $\mathcal{T}$ (the *test iterations*). Of course, as in usual trust-region methods, we expect to use this information when available, and we therefore impose that, whenever $k \in \mathcal{T}$ and $\chi_k < 0$,

$$m_k(x_k) - m_k(x_k + s_k) \geq \kappa_{\text{wcc}} |\chi_k| \min[\chi_k^2, \Delta_k^2] \tag{2.8}$$

for some constant $\kappa_{\text{wcc}} \in (0, \frac{1}{2})$, which is a weak version of (2.6). We call this condition the *weak curvature condition*. Given this limited curvature information, it is unrealistic to expect convergence to points satisfying the usual second-order necessary optimality conditions. However, the definition of χ_k suggests that we may obtain convergence to *weakly second-order critical points* (with respect to the directions in $\mathcal{D}$). Such a point x_* is defined by the property that

$$\nabla_x f(x_*) = 0 \;\text{ and }\; \chi(x_*) \stackrel{\text{def}}{=} \min_{d \in \mathcal{D}} \langle d, \nabla_{xx} f(x_*) d \rangle \geq 0. \tag{2.9}$$

But what happens when χ_k is not known ($k \notin \mathcal{T}$)? Are we then free to choose just any step satisfying the Cauchy condition (2.4)? Our analysis shows that the *quadratic model decrease* condition

$$m_k(x_k) - m_k(x_k + s_k) \geq \kappa_{\text{qmd}} \|s_k\|^2 \tag{2.10}$$

(for some constant $\kappa_{\text{qmd}} > 0$) is all we need to impose for $k \notin \mathcal{T}$ in order to guarantee convergence to weakly second-order critical points. As we will show in Lemma 2.1 below, this easily verifiable condition is automatically satisfied if the model is quadratic and the curvature along the step s_k safely positive. It is related to the step-size rule discussed in the linesearch framework in [4], [6] and [7], where the step is also restricted to meet a quadratic decrease condition on the value of the objective function. In practice, the choice of the step s_k may be implemented as a two stage process, as follows. If χ_k is known at iteration k, one computes a step in the trust region satisfying both (2.4) and, if $\chi_k < 0$, (2.8), thus exploiting negative curvature if present. If χ_k is unknown, one first computes a step in the trust region satisfying (2.4). If it also satisfies (2.10), we may then proceed. Otherwise, the value of χ_k is computed (forcing the inclusion $k \in \mathcal{T}$) and a new step is recomputed in the trust region, which exploits any detected negative curvature in that it satisfies both (2.4) and (2.8) (if $\chi_k < 0$).

We now incorporate all this discussion in the formal statement of our trust-region algorithm with incomplete curvature information, which is shown as Algorithm 2.1 on page 5.

Algorithm 2.1: Algorithm with Incomplete Curvature Information

Step 0: Initialization. An initial point x_0 and an initial trust-region radius Δ_0 are given. The constants η_1, η_2, γ_1, γ_2, γ_3 and γ_4 are also given and satisfy $0 < \eta_1 \leq \eta_2 < 1$ and $0 < \gamma_1 \leq \gamma_2 < 1 \leq \gamma_3 \leq \gamma_4$. Compute $f(x_0)$ and set $k = 0$.

Step 1: Step calculation. Compute a step s_k satisfying the Cauchy condition (2.4) with $\|s_k\| \leq \Delta_k$ and such that either χ_k is known and the weak curvature condition (2.8) holds if $\chi_k < 0$, or the quadratic model decrease condition (2.10) holds.

Step 2: Acceptance of the trial point. Compute $f(x_k + s_k)$ and define ρ_k by (2.1). If $\rho_k \geq \eta_1$, then define $x_{k+1} = x_k + s_k$; otherwise define $x_{k+1} = x_k$.

Step 3: Trust-region radius update. Set

$$\Delta_{k+1} \in \begin{cases} [\gamma_3\Delta_k, \gamma_4\Delta_k] & \text{if } \rho_k \geq \eta_2, \\ [\gamma_2\Delta_k, \Delta_k] & \text{if } \rho_k \in [\eta_1, \eta_2), \\ [\gamma_1\Delta_k, \gamma_2\Delta_k] & \text{if } \rho_k < \eta_1. \end{cases} \tag{2.11}$$

Increment k by one and go to Step 1.

As is usual with trust-region methods, we say that iteration k is *successful* when $\rho_k \geq \eta_1$ and *very successful* when $\rho_k \geq \eta_2$. The set $\mathcal{S}$ is defined as the index set of all successful iterations.

We now clarify the relation between (2.10) and the curvature along the step, and show that every step with significant curvature automatically satisfies (2.10).

Lemma 2.1 *Suppose that the quadratic model (2.2) is used. Suppose also that, for some $\epsilon > 0$, $\langle s_k, H_k s_k\rangle \geq \epsilon\|s_k\|^2$ and that the minimum of $m_k(x_k + \alpha s_k)$ occurs for $\alpha = \alpha_* \geq 1$. Then (2.10) holds with $\kappa_{qmd} = \frac{1}{2}\epsilon$.*

Proof. Our assumptions imply that $\langle g_k, s_k\rangle < 0$ and that $m_k(x_k + \alpha s_k)$ is a convex quadratic function in the parameter α, with

$$\alpha_* = \frac{|\langle g_k, s_k\rangle|}{\langle s_k, H_k s_k\rangle} \quad \text{and} \quad m_k(x_k + \alpha_* s_k) = f(x_k) - \tfrac{1}{2}\alpha_*|\langle g_k, s_k\rangle|.$$

Thus,

$$m_k(x_k) - m_k(x_k + \alpha_* s_k) = \tfrac{1}{2}\alpha_*|\langle g_k, s_k\rangle| = \kappa_k\|\alpha_* s_k\|^2,$$

where κ_k is given by

$$\kappa_k = \frac{|\langle g_k, s_k\rangle|}{2\alpha_*\|s_k\|^2} = \frac{\langle s_k, H_k s_k\rangle}{2\|s_k\|^2} \geq \tfrac{1}{2}\epsilon.$$

As a consequence, the convex quadratic $m_k(x_k + \alpha s_k)$ and the concave quadratic $m_k(x_k) - \kappa_k \alpha^2 \|s_k\|^2$ coincide at $\alpha = 0$ and $\alpha = \alpha_*$. The value of the former is thus smaller than that of the latter for $\alpha = 1 \in [0, \alpha_*]$, giving

$$m_k(x_k + s_k) \le m_k(x_k) - \kappa_k \|s_k\|^2 \le m_k(x_k) - \tfrac{1}{2}\epsilon \|s_k\|^2.$$

Hence (2.10) holds with $\kappa_{\rm qmd} = \frac{1}{2}\epsilon$, as requested. □

Note that the assumption $\alpha_* \ge 1$ simply states that the step s_k does not extend beyond the one-dimensional minimizer in the direction $s_k/\|s_k\|$. This is a very reasonable requirement, which is, for instance, automatically satisfied if the step is computed by a truncated conjugate-gradient algorithm.

3 Asymptotic second-order properties

We investigate in this section the convergence properties of Algorithm 2.1. We first specify our assumptions, which we separate between assumptions on the objective function of (1.1), assumptions on the model m_k and assumptions on the algorithm.

Our assumptions on the objective function are identical to those in [2, Chapter 6]:

AF.1 $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on $\mathbb{R}^n$.

AF.2 The objective function is bounded below, that is, $f(x) \ge \kappa_{\rm lbf}$ for all $x \in \mathbb{R}^n$ and some constant $\kappa_{\rm lbf}$.

AF.3 The Hessian of f is uniformly bounded, that is, $\|\nabla_{xx} f(x)\| \le \kappa_{\rm uH} - 1$ for all $x \in \mathbb{R}^n$ and some constant $\kappa_{\rm uH} > 1$.

AF.4 The Hessian of f is Lipschitz continuous with constant $\kappa_{\rm LH}$ on $\mathbb{R}^n$, that is, there exists a constant $\kappa_{\rm LH} > 0$ such that

$$\|\nabla_{xx} f(x) - \nabla_{xx} f(y)\| \le \kappa_{\rm LH} \|x - y\| \;\text{ for all }\; x, y \in \mathbb{R}^n.$$

Beyond our requirement that the model and objective function coincide to first-order, our assumptions on the model reduce to the following conditions:

AM.1 The Hessian of m_k is uniformly bounded, that is, $\|\nabla_{xx} m_k(x)\| \le \kappa_{\rm uH} - 1$ for all $x \in \mathcal{B}_k$ and all k (possibly choosing a larger value of $\kappa_{\rm uH}$).

AM.2 The Hessian of m_k is Lipschitz continuous, that is,

$$\|\nabla_{xx} m_k(x) - \nabla_{xx} m_k(y)\| \le \kappa_{\rm LH} \|x - y\| \;\text{ for all }\; x, y \in \mathcal{B}_k \;\text{ and all }\; k$$

(possibly choosing a larger value of $\kappa_{\rm LH}$).

AM.3 The model's Hessian at x_k coincides with the true Hessian along the test directions and the current step whenever a first-order critical point is approached, that is,

$$\lim_{k\to\infty} \left[\max_{d \in \mathcal{D} \cup \{s_k/\|s_k\|\}} |\langle d, [\nabla_{xx} f(x_k) - H_k] d\rangle| \right] = 0 \;\text{ when }\; \lim_{k\to\infty} \|g_k\| = 0.$$

Note that this condition obvioulsy holds in the frequent case where (2.2) holds and $H_k = \nabla_{xx} f(x_k)$. Finally, we make the following assumptions on the algorithm:

AA.1 If $k \in \mathcal{T} \setminus \mathcal{S}$, then the next successful iteration after k (if any) belongs to $\mathcal{T}$.

This requirement is motivated by our desire to use negative curvature information when available. Thus, if negative curvature is detected in the neighbourhood of x_k, we impose that it cannot be ignored when moving to another iterate. If we choose $H_k = \nabla_{xx} f(x_k)$, then obviously $\chi_{k+j} = \chi_k$ as long as iteration $k+j-1$ is unsuccessful ($j \geq 1$), and AA.1 automatically holds. This is also the case if we choose not to update Hessian approximations at unsuccessful iterations, a very common strategy.

AA.2 There are infinitely many test iterations, that is, $|\mathcal{T}| = +\infty$.

If AA.2 does not hold, then no information on curvature is known for large enough k, and there is no reason to expect convergence to a point satisfying second-order necessary conditions.

We start our convergence analysis by noting that the stated assumptions are sufficient for ensuring first-order convergence. We state the corresponding result (extracted from [2, Theorem 6.4.6]) for future reference.

Theorem 3.1 *Suppose that AF.1–AF.3 and AM.1 hold. Then (2.5) holds.*

Note that only the boundedness of the model's and objective function's Hessian is needed for this result. We next provide a slight reformulation of Lemma 6.5.3 in [2], which we explicitly need below. This lemma states that the iterations must be asymptotically very successful when the steps are sufficiently small and a quadratic model decrease condition similar to (2.10) holds.

Lemma 3.2 *Suppose that AF.1–AF.4 and AM.1–AM.3 hold. Then, for any $\kappa > 0$, there exists a $k_1(\kappa) \geq 0$ such that, if $k \geq k_1(\kappa)$,*

$$m_k(x_k) - m_k(x_k + s_k) \geq \kappa \|s_k\|^2 \tag{3.1}$$

and

$$\|s_k\| \leq \delta_1(\kappa) \stackrel{\text{def}}{=} \frac{(1-\eta_2)\kappa}{2\kappa_{LH}}, \tag{3.2}$$

then $\rho_k \geq \eta_2$.

Proof. We first use Theorem 3.1 and AM.3 to deduce the existence of a $k_1(\kappa) \geq 0$ such that

$$|\langle \frac{s_k}{\|s_k\|}, [\nabla_{xx} f(x_k) - H_k] \frac{s_k}{\|s_k\|} \rangle| \leq (1-\eta_2)\kappa \tag{3.3}$$

for $k \geq k_1(\kappa)$. Assuming now that $k \geq k_1(\kappa)$ and that (3.1) and (3.2) hold, we obtain from the mean-value theorem that, for all $k \geq k_1(\kappa)$ and for some ξ_k and ζ_k

in the segment $[x_k, x_k + s_k]$,

$$\begin{array}{rcl} |\rho_k - 1| & = & \left|\dfrac{f(x_k+s_k) - m_k(x_k+s_k)}{m_k(x_k) - m_k(x_k+s_k)}\right| \\ & \leq & \dfrac{1}{2\kappa\|s_k\|^2}|\langle s_k, \nabla_{xx}f(\xi_k)s_k\rangle - \langle s_k, \nabla_{xx}m_k(\zeta_k)s_k\rangle| \\ & = & \dfrac{1}{2\kappa\|s_k\|^2}|\langle s_k, [\nabla_{xx}f(\xi_k) - \nabla_{xx}m_k(\zeta_k)]s_k\rangle| \\ & \leq & \dfrac{1}{2\kappa}\Big[\|\nabla_{xx}f(\xi_k) - \nabla_{xx}f(x_k)\| + |\langle \frac{s_k}{\|s_k\|}, [\nabla_{xx}f(x_k) - H_k]\frac{s_k}{\|s_k\|}\rangle| \\ & & \qquad\qquad +\|\nabla_{xx}m_k(\zeta_k) - H_k\|\Big], \end{array} \tag{3.4}$$

where we also used (3.1), the triangle inequality and the Cauchy-Schwarz inequality. By AF.4, AM.2 and the bounds $\|\xi_k - x_k\| \leq \|s_k\|$ and $\|\zeta_k - x_k\| \leq \|s_k\|$, the inequality (3.4) becomes

$$|\rho_k - 1| \leq \frac{1}{\kappa}\Big[\kappa_{\rm LH}\|s_k\| + \frac{1}{2}|\langle \frac{s_k}{\|s_k\|}, [\nabla_{xx}f(x_k) - H_k]\frac{s_k}{\|s_k\|}\rangle|\Big].$$

The conclusion of the lemma then follows from (3.2) and (3.3). □

We now prove that an iteration at which the step is sufficiently small must be asymptotically very successful if the curvature of the objective function along a test direction is negative.

Lemma 3.3 *Suppose that AF.1–AF.4 and AM.1–AM.3 hold. Suppose also that the condition $\chi(x_{k_i}) \leq \chi_*$ is satisfied for some infinite subsequence $\{k_i\}$ and some constant $\chi_* < 0$. Then there exist a $k_2(|\chi_*|) \geq 0$ and a $\delta_2(|\chi_*|) > 0$ such that, if $k_i \geq k_2(|\chi_*|)$ and $0 < \|s_{k_i}\| \leq \delta_2(|\chi_*|)$, then $\rho_{k_i} \geq \eta_2$.*

Proof. Observe first that AM.3, Theorem 3.1 and the inequality $\chi(x_{k_i}) \leq \chi_*$ ensure that, for some $k_c \geq 0$ sufficiently large,

$$\chi_{k_i} \leq \tfrac{1}{2}\chi_* < 0$$

for $k_i \geq k_c$. Suppose now that

$$\|s_{k_i}\| \leq \tfrac{1}{2}|\chi_*| \tag{3.5}$$

and consider first a $k_i \geq k_c$ with $k_i \in \mathcal{T}$. In this case, the algorithm ensures that (2.8) holds at iteration k_i, and therefore that

$$\begin{array}{rcl} m_{k_i}(x_{k_i}) - m_{k_i}(x_{k_i} + s_{k_i}) & \geq & \frac{1}{2}\kappa_{\rm wcc}|\chi_*| \, \min[\frac{1}{4}\chi_*^2, \Delta_{k_i}^2] \\ & \geq & \frac{1}{2}\kappa_{\rm wcc}|\chi_*| \, \min[\frac{1}{4}\chi_*^2, \|s_{k_i}\|^2] \\ & = & \frac{1}{2}\kappa_{\rm wcc}|\chi_*| \, \|s_{k_i}\|^2, \end{array} \tag{3.6}$$

where we used the inequality $\|s_{k_i}\| \leq \Delta_{k_i}$ to derive the second inequality and (3.5) to derive the last. On the other hand, if $k_i \notin \mathcal{T}$, then (2.10) holds, which, together with (3.6), gives that

$$m_{k_i}(x_{k_i}) - m_{k_i}(x_{k_i} + s_{k_i}) \geq \min[\kappa_{\rm qmd}, \tfrac{1}{2}\kappa_{\rm wcc}|\chi_*|] \, \|s_{k_i}\|^2.$$

This inequality now allows us to apply Lemma 3.2 with

$$\kappa = \min[\kappa_{\rm qmd}, \tfrac{1}{2}\kappa_{\rm wcc}|\chi_*|] \stackrel{\rm def}{=} \kappa(|\chi_*|)$$

for each $k_i \geq k_c$ such that (3.5) holds. We then deduce that, if

$$k_i \geq \max[k_c, k_1(\kappa(|\chi_*|))] \stackrel{\rm def}{=} k_2(|\chi_*|)$$

and

$$0 < \|s_{k_i}\| \leq \min[\tfrac{1}{2}|\chi_*|, \delta_1(\kappa(|\chi_*|))] \stackrel{\rm def}{=} \delta_2(|\chi_*|),$$

then $\rho_{k_i} \geq \eta_2$, which completes the proof. □

We next proceed to examine the nature of the critical points to which Algorithm 2.1 is converging under our assumptions. In what follows, we essentially adapt the results of [2, Section 6.6.3] in our context of incomplete curvature information.

We first consider the case where the number of successful iterations is finite.

Theorem 3.4 *Suppose that AF.1–AF.4 and AM.1–AM.3 hold. Suppose also that $|\mathcal{S}| < +\infty$. Then the unique limit point x_* of the sequence of iterates is weakly second-order critical.*

Proof. We first note that Theorem 3.1 guarantees that x_* is a first-order critical point. Moreover, since $|\mathcal{S}|$ is finite, there exists a last successful iteration $k_s \geq 0$, and therefore $x_k = x_{k_s+1} = x_*$ for all $k > k_s$. But the mechanism of the algorithm then ensures that

$$\lim_{k\to\infty} \Delta_k = 0. \tag{3.7}$$

For the purpose of obtaining a contradiction, we now suppose that $\chi(x_*) < 0$. This implies that $\chi(x_k) = \chi(x_*) < 0$ for all $k > k_s$. Applying now Lemma 3.3 to the subsequence $\{k \mid k > k_s\}$ and taking (3.7) into account, we then deduce that, for k sufficiently large, $\rho_k \geq \eta_2$ and thus $k \in \mathcal{S}$. This is impossible since $|\mathcal{S}| < +\infty$. Hence $\chi(x_*) \geq 0$, and the proof is complete. □

Having proved the desired result for $|\mathcal{S}| < +\infty$, we now assume, for the rest of this section, that $|\mathcal{S}| = +\infty$. Combining this assumption with AA.1 and AA.2 then yields the following result.

Lemma 3.5 *Suppose that AA.1–AA.2 hold and that $|\mathcal{S}| = +\infty$. Then*

$$|\mathcal{S} \cap \mathcal{T}| = +\infty. \tag{3.8}$$

Moreover, for any infinite subsequence $\{k_i\} \subseteq \mathcal{T}$, there exists an infinite subsequence $\{k_j\} \subseteq \mathcal{S} \cap \mathcal{T}$ such that $\{x_{k_j}\} \subseteq \{x_{k_i}\}$.

Proof. If $\mathcal{S} \cap \mathcal{T}$ is finite, then, since $\mathcal{T}$ is infinite because of AA.2, all test iterations $k \in \mathcal{T}$ sufficiently large must be unsuccessful, which is impossible because of AA.1 and the infinite nature of $\mathcal{S}$. Hence (3.8) holds. Consider now an infinite subsequence $\{k_i\} \subseteq \mathcal{T}$. The fact that $\mathcal{S}$ is infinite ensures that there exists a successful iteration $k_i + p_i$ ($p_i \geq 0$) after k_i such that $x_{k_i+p_i} = x_{k_i}$. Assumption AA.1 then guarantees that $k_i + p_i \in \mathcal{S} \cap \mathcal{T}$. Hence the desired conclusion follows with $\{k_j\} = \{k_i + p_i\}$. □

In this context, we first show that the curvature of the objective function is asymptotically non-negative along the test directions in $\mathcal{D}$, at least on some subsequence.

Theorem 3.6 (Theorem 6.6.4 in [2]) *Suppose that AF.1–AF.4, AM.1–AM.3 and AA.1–AA.2 hold. Then*

$$\limsup_{k\to\infty} \chi(x_k) \geq 0.$$

Proof. Assume, for the purpose of deriving a contradiction, that there is a constant $\chi_* < 0$ such that, for all k,

$$\chi(x_k) \leq \chi_*. \tag{3.9}$$

Observe first that this bound, AM.3 and Theorem 3.1 imply together that, for some $k_\chi \geq 0$ sufficiently large,

$$\chi_k \leq \tfrac{1}{2}\chi_* < 0 \tag{3.10}$$

for all $k \geq k_\chi$. On the other hand, (3.9), the bound $\|s_k\| \leq \Delta_k$ and Lemma 3.3 applied to the subsequence $\{k \geq k_\chi\}$ ensure that

$$\rho_k \geq \eta_2 \;\text{ for all }\; k \geq k_* \stackrel{\rm def}{=} \max[k_2(|\chi_*|), k_\chi] \;\text{ such that }\; \Delta_k \leq \delta_2(|\chi_*|).$$

Thus, each iteration k such that these conditions hold must be very successful and the mechanism of the algorithm then ensures that $\Delta_{k+1} \geq \Delta_k$. As a consequence, we obtain that, for all $j \geq 0$,

$$\Delta_{k_*+j} \geq \min[\gamma_1\delta_2(|\chi_*|), \Delta_{k_*}] \stackrel{\rm def}{=} \delta_*. \tag{3.11}$$

Now consider $\mathcal{K} = \{k \in \mathcal{S} \cap \mathcal{T} \mid k \geq k_*\}$. For each $k \in \mathcal{K}$, we then have that

$$f(x_k) - f(x_{k+1}) \geq \eta_1[m_k(x_k) - m_k(x_k+s_k)] \geq \tfrac{1}{2}\eta_1\kappa_{\rm wcc}|\chi_*|\min[\tfrac{1}{4}\chi_*^2, \delta_*^2] > 0, \tag{3.12}$$

where the first inequality results from $k \in \mathcal{S}$, and the second from $k \in \mathcal{T}$, (2.8), (3.10) and (3.11). Since $|\mathcal{K}|$ is infinite because of (3.8), this is impossible as (3.12) and the non-increasing nature of the sequence $\{f(x_k)\}$ imply that $f(\cdot)$ is unbounded below, in contradiction with AF.2. Hence (3.9) cannot hold and the proof of the theorem is complete. □

Observe that this theorem does not make the assumption that the sequence of iterates has limit points. If we now assume that the sequence of iterates is bounded, then limit points exist and are finite, and we next show that at least one of them is weakly second-order critical.

Theorem 3.7 (Theorem 6.6.5 in [2]) *Suppose that AF.1–AF.4, AM.1–AM.3 and AA.1–AA.2 hold. Suppose also that the sequence $\{x_k\}$ lies within a closed, bounded domain. Then there exists at least one limit point x_* of $\{x_k\}$ which is weakly second-order critical.*

Proof. Theorem 3.6 ensures that there is a subsequence of iterates $\{x_{k_i}\}$ such that

$$\lim_{i\to\infty} \chi(x_{k_i}) \geq 0.$$

Since this sequence remains in a compact domain, it must have at a limit point x_*. We then deduce from the continuity of $\chi(\cdot)$ that

$$\chi(x_*) \geq 0 \;\text{ and }\; \nabla_x f(x_*) = 0,$$

where the equality follows from Theorem 3.1. □

To obtain further useful results, we prove the following technical lemma.

Lemma 3.8 (Lemma 6.6.6 in [2]) *Suppose that AF.1–AF.4, AM.1–AM.3 and AA.1–AA.2 hold. Suppose also that x_* is a limit point of a sequence of iterates and that $\chi(x_*) < 0$. Then, for every infinite subsequence $\{x_{k_i}\}$ of iterates such that $\{k_i\} \subseteq \mathcal{T}$ and $\chi(x_{k_i}) \leq \frac{1}{2}\chi(x_*)$, we have that*

$$\lim_{i\to\infty} \Delta_{k_i} = 0. \tag{3.13}$$

Moreover, (3.13) also holds for any subsequence $\{x_{k_i}\}$ of iterates converging to x_* and such that $\{k_i\} \subseteq \mathcal{T}$.

Proof. Define $\chi_* \stackrel{\text{def}}{=} \chi(x_*)$. Consider a subsequence $\{x_{k_i}\}$ such that $\{k_i\} \subseteq \mathcal{T}$ and $\chi(x_{k_i}) \leq \frac{1}{2}\chi_*$. Because of Lemma 3.5, we may assume without loss of generality that $\{k_i\} \subseteq \mathcal{S} \cap \mathcal{T}$. Moreover, Theorem 3.1 ensures that $\|g_{k_i}\|$ converges to zero, which, combined with AM.3, gives that

$$\chi_{k_i} \leq \tfrac{1}{2}\chi(x_{k_i}) \leq \tfrac{1}{4}\chi_* < 0 \tag{3.14}$$

holds for i sufficiently large. Now suppose, for the purpose of deriving a contradiction, that there exists an $\epsilon \in (0,1)$ such that

$$\Delta_{k_i} \geq \epsilon \tag{3.15}$$

for all i. Successively using the inclusion $k_i \in \mathcal{S} \cap \mathcal{T}$, (2.8), (3.14) and (3.15), we then obtain that, for all i sufficiently large,

$$f(x_{k_i}) - f(x_{k_i+1}) \geq \eta_1[m_k(x_{k_i}) - m_{k_i}(x_{k_i} + s_{k_i})] \geq \tfrac{1}{4}\eta_1\kappa_{\text{wcc}}|\chi_*| \min[\tfrac{1}{16}\chi_*^2, \epsilon^2] > 0.$$

Because of (3.8) and because the sequence $\{f(x_k)\}$ is non-increasing, this is impossible since it would imply that $f(\cdot)$ is unbounded below, in contradiction with AF.2. As a consequence, we obtain that Δ_{k_i} converges to zero, which is the first conclusion of the lemma. The second conclusion also holds because, if $\{x_{k_i}\}$ converges to x_*, AF.1 and the continuity of $\chi(\cdot)$ imply that $\chi(x_{k_i})$ converges to $\chi(x_*)$ and therefore that $\chi(x_{k_i}) \leq \frac{1}{2}\chi(x_*)$ for all i sufficiently large. The limit (3.13) then follows by applying the first part of the lemma. □

In the usual trust-region framework, where every iteration is a test iteration, a variant of Theorem 3.7 can still be proved if one no longer assumes that the iterates remain in a bounded set, but rather considers the case of an isolated limit point. This result may be generalized to our case, at the price of the following additional assumption.

AA.3 There exists an integer $m \geq 0$ such that the number of successful iterations between successive indices in $\mathcal{T}$ does not exceed m.

This assumption requires that the set of test iterations at which χ_k is known and negative curvature possibly exploited is not too sparse in the set of all iterations. In other words, the presence of negative curvature should be investigated often enough. Useful consequences of AA.3 are given by the following lemma.

Lemma 3.9 *Suppose that assumptions AF.1–AF.4, AM.1–AM.3 and AA.1–AA.3 hold. Then there exists a constant* $\kappa_{\rm dst} > 1$ *such that, if* $k \in \mathcal{S} \cap \mathcal{T}$*, then*

$$\Delta_{k+j} \leq \gamma_4^m \Delta_k \tag{3.16}$$

and

$$\|x_{k+j} - x_k\| \leq \kappa_{\rm dst} \Delta_k \tag{3.17}$$

for all $0 \leq j \leq p$*, where* $k+p$ *is the index immediately following* k *in* $\mathcal{S} \cap \mathcal{T}$*.*

Proof. We first note that the mechanism of the algorithm imposes that $\Delta_{\ell_+} \leq \gamma_4 \Delta_\ell$ for all $\ell \in \mathcal{S}$, where ℓ_+ is the first successful iteration after ℓ. Let $k \in \mathcal{S} \cap \mathcal{T}$ and $k+p$ be the index immediately following k in $\mathcal{S} \cap \mathcal{T}$. Hence, if we consider $k \leq \ell \leq k+p$ and define $q(\ell) \stackrel{\rm def}{=} |\mathcal{S} \cap \{k, \ldots, \ell\}|$, that is the number of successful iterations from k to ℓ (including k), we obtain from AA.3 that

$$q(\ell) \leq m + 1 \tag{3.18}$$

and thus that

$$\Delta_\ell \leq \gamma_4^{q(\ell)-1} \Delta_k \leq \gamma_4^m \Delta_k,$$

which proves (3.16). Now, because the trial point is only accepted at successful iterations, we have that, for $0 \leq j \leq p$,

$$\|x_{k+j} - x_k\| \leq \sum_{\ell=k}^{k+j-1}{}^{(\mathcal{S})} \|x_{\ell+1} - x_\ell\| \leq \sum_{\ell=k}^{k+j-1}{}^{(\mathcal{S})} \Delta_\ell$$

where the sum superscripted by $(\mathcal{S})$ is restricted to the successful iterations. Using this inequality, (3.18) and (3.16), we then obtain (3.17) with $\kappa_{\rm dst} = \max[1, m\gamma_4^m]$. □

We are now in position to assess the nature of an isolated limit point of a subsequence of test iterations.

Theorem 3.10 (Theorem 6.6.7 in [2]) *Suppose that AF.1–AF.4, AM.1–AM.3 and AA.1–AA.3 hold. Suppose also that x_* is an isolated limit point of the sequence of iterates $\{x_k\}$ and that there exists a subsequence $\{k_i\} \subseteq \mathcal{T}$ such that $\{x_{k_i}\}$ converges to x_*. Then x_* is a weakly second-order critical point.*

Proof. Let x_* and $\{x_{k_i}\}$ satisfy the theorem's assumptions, and note that, because of Lemma 3.5, we may suppose without loss of generality that $\{k_i\} \subseteq \mathcal{S} \cap \mathcal{T}$. Now assume, for the purpose of deriving a contradiction, that $\chi(x_*) < 0$. We may then apply the second part of Lemma 3.8 to the subsequence $\{k_i\}$ and deduce that

$$\lim_{i \to \infty} \Delta_{k_i} = 0. \tag{3.19}$$

But, since x_* is isolated, there must exist a $\varepsilon > 0$ such that any other limit point of the sequence $\{x_k\}$ is at a distance at least ε from x_*. Moreover, we have that, for each x_k with k sufficiently large, either

$$\|x_k - x_*\| \leq \tfrac{1}{8}\varepsilon \;\text{ or }\; \|x_k - x_*\| \geq \tfrac{1}{2}\varepsilon. \tag{3.20}$$

In particular,

$$\|x_{k_i} - x_*\| \leq \tfrac{1}{8}\varepsilon$$

for i large enough. Combining this last bound, Lemma 3.9 and (3.19), we see that, for all $0 \leq j \leq p$,

$$\|x_{k_i+j} - x_*\| \leq \|x_{k_i} - x_*\| + \|x_{k_i+j} - x_{k_i}\| \leq \|x_{k_i} - x_*\| + \kappa_{\rm dst}\Delta_{k_i} \leq \tfrac{1}{8}\varepsilon + \tfrac{1}{8}\varepsilon = \tfrac{1}{4}\varepsilon$$

for i sufficiently large, where $k_i + p$ is, as above, the index immediately following k_i in $\mathcal{S} \cap \mathcal{T}$. As a consequence, (3.20) implies that

$$\|x_{k_i+j} - x_*\| \leq \tfrac{1}{8}\varepsilon$$

for i sufficiently large and all j between 0 and p. Applying this argument repeatedly, which is possible because of (3.8), we obtain that the complete sequence $\{x_k\}$ converges to x_*. We may therefore apply the second part of Lemma 3.8 to the subsequence $\mathcal{S} \cap \mathcal{T}$ and deduce that

$$\lim_{\substack{k \to \infty \\ k \in \mathcal{S} \cap \mathcal{T}}} \Delta_k = 0.$$

But the bound (3.16) in Lemma 3.9 then implies that

$$\lim_{k \to \infty} \Delta_k = 0, \tag{3.21}$$

and thus that

$$\lim_{k \to \infty} \|s_k\| = 0. \tag{3.22}$$

Moreover, AF.1, Theorem 3.1 and AM.3 ensure that $\chi(x_k) \le \frac{1}{2}\chi(x_*) < 0$ for all k sufficiently large. We may now use this bound and the limit (3.22) to apply Lemma 3.3 to the subsequence of all sufficiently large $\{k\}$, which then yields that all iterations are eventually very successful and hence that the trust-region radii are bounded away from zero. But this is impossible in view of (3.21). Our initial assumption must therefore be false, and $\chi(x_*) \ge 0$, which is enough to conclude the proof because of Theorem 3.1. $\square$

Remarkably, introducing an additional but realistic assumption allows us to strengthen these convergence results. The new assumption simply imposes that the trust-region radius *increases* at very successful iterations. This is formally expressed as:

AA.4 $\gamma_3 > 1$.

We now state our final and strongest convergence result, which avoids the assumption that limit points are isolated.

Theorem 3.11 (Theorem 6.6.8 in [2]) *Suppose that AF.1–AF.4, AM.1–AM.3 and AA.1–AA.4 hold. Suppose also that x_* is a limit point of the sequence of iterates $\{x_k\}$ and that there exists a subsequence $\{k_i\} \subseteq \mathcal{T}$ such that $\{x_{k_i}\}$ converges to x_*. Then x_* is a weakly second-order critical point.*

Proof. As above, Lemma 3.5 allows us to restrict our attention to subsequences in $\mathcal{S} \cap \mathcal{T}$ without loss of generality. The desired conclusion will then follow if we prove that every limit point of the iterates indexed by $\mathcal{S} \cap \mathcal{T}$ is weakly second-order critical. Let x_* be any such limit point and assume, again for the purpose of deriving a contradiction, that $\chi_* \stackrel{\text{def}}{=} \chi(x_*) < 0$. We first note that AF.1 and the continuity of $\chi(\cdot)$ imply that, for some $\delta_{\text{out}} > 0$,

$$\chi(x) \le \tfrac{1}{2}\chi_* < 0 \;\text{ whenever }\; x \in \mathcal{V}_{\text{out}}, \tag{3.23}$$

where $\mathcal{V}_{\text{out}} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \|x - x_*\| \le \delta_{\text{out}}\}$ is an (outer) neighbourhood of x_*. Now define $\mathcal{K}_{\text{out}} = \mathcal{S} \cap \mathcal{T} \cap \{k \,|\, x_k \in \mathcal{V}_{\text{out}}\}$. Observe that, because of (3.23),

$$\chi(x_k) \le \tfrac{1}{2}\chi_* < 0 \tag{3.24}$$

for all $k \in \mathcal{K}_{\text{out}}$. We may then apply the first part of Lemma 3.8 to the subsequence $\mathcal{K}_{\text{out}}$ and deduce that

$$\lim_{\substack{k\to\infty \\ k \in \mathcal{K}_{\text{out}}}} \Delta_k = 0. \tag{3.25}$$

We now define a smaller (inner) neighbourhood of x_* by

$$\mathcal{V}_{\text{in}} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \|x - x_*\| \le \delta_{\text{in}}\},$$

where we choose

$$\delta_{\text{in}} < \frac{\delta_{\text{out}}}{1 + \kappa_{\text{dst}}} \le \frac{\delta_{\text{out}}}{2}, \tag{3.26}$$

and consider an iterate $x_\ell \in \mathcal{V}_{\rm in}$ such that $\ell \in \mathcal{S} \cap \mathcal{T}$. Then either the subsequence $\{x_{\ell+j}\}$ remains in $\mathcal{V}_{\rm out}$ for all j sufficiently large or it leaves this neighbourhood.

Consider first the case where $\{x_{\ell+j}\}$ remains in $\mathcal{V}_{\rm out}$ for all j large enough. Since $\mathcal{K}_{\rm out}$ and $\mathcal{S} \cap \mathcal{T}$ coincide for large enough indices in this case, we obtain from (3.25) that

$$\lim_{\substack{k\to\infty \\ k\in\mathcal{S}\cap\mathcal{T}}} \Delta_k = 0.$$

As above, the bound (3.16) in Lemma 3.9 then gives that

$$\lim_{k\to\infty} \Delta_k = 0, \tag{3.27}$$

and thus that $\lim_{k\to\infty} \|s_k\| = 0$. We may now use this last limit and the bound (3.23) at x_k to apply Lemma 3.3 to the subsequence of all sufficiently large $\{k\}$, and deduce that all iterations are eventually very successful. Hence the trust-region radii are bounded away from zero. But this is impossible in view of (3.27). The subsequence $\{x_{\ell+j}\}$ must therefore leave the outer neighbourhood $\mathcal{V}_{\rm out}$.

As a consequence, and since x_* is a limit point of the subsequence of iterates indexed by $\mathcal{S} \cap \mathcal{T}$, there must exist a subsequence of iterates $\{x_{k_s}\}$ such that for all $s \geq 0$, $x_{k_s} \in \mathcal{V}_{\rm in}$, $k_s \in \mathcal{S} \cap \mathcal{T}$,

$$\Delta_{k_s} \leq \delta_{\rm in}, \tag{3.28}$$

and there exists an iterate x_{q_s+1} with $k_s < q_s + 1 < k_{s+1}$ such that x_{q_s+1} is the first iterate after x_{k_s} not belonging to $\mathcal{V}_{\rm out}$. (Note that (3.28) is possible because of (3.25) and $\mathcal{V}_{\rm in} \subseteq \mathcal{V}_{\rm out}$.) Now let x_{r_s} be the iterate whose index immediately follows k_s in the sequence $\mathcal{S} \cap \mathcal{T}$. From Lemma 3.9 and (3.26), we obtain that

$$\|x_{r_s} - x_*\| \leq \|x_{k_s} - x_*\| + \|x_{r_s} - x_{k_s}\| \leq \|x_{k_s} - x_*\| + \kappa_{\rm dst}\Delta_{k_s} \leq \delta_{\rm in} + \kappa_{\rm dst}\delta_{\rm in} < \delta_{\rm out},$$

and thus x_{r_s} must belong to $\mathcal{V}_{\rm out}$. If we finally define x_{t_s} to be the last successful test iterate before x_{q_s+1}, i.e., the largest index not exceeding q_s such that $t_s \in \mathcal{S}\cap\mathcal{T}$, then we must have that

$$k_s < r_s \leq t_s \leq q_s < q_s + 1 < k_{s+1}$$

and $x_{t_s} \in \mathcal{V}_{\rm out}$. Moreover, Lemma 3.9 applied for $k = t_s$ and the definition of t_s then ensure that

$$\Delta_{q_s} \leq \gamma_4^m \Delta_{t_s}. \tag{3.29}$$

If we let the subsequence $\mathcal{K}_*$ to be defined by

$$\mathcal{K}_* \stackrel{\rm def}{=} \{k \mid k_s \leq k \leq q_s \;\text{ for all }\; s \geq 0\},$$

we have that $t_s \in \mathcal{S} \cap \mathcal{T} \cap \mathcal{K}_*$ and $x_k \in \mathcal{V}_{\rm out}$ for all $k \in \mathcal{K}_*$. Using (3.23), we may then apply Lemma 3.3 to $\mathcal{K}_*$ and deduce that there exist a $k_* \stackrel{\rm def}{=} k_2(|\chi_*|) \geq 0$ and a $\delta_* \stackrel{\rm def}{=} \delta_2(|\chi_*|) \in (0, \delta_{\rm out}]$ such that

$$\rho_k \geq \eta_2 \;\text{ for all }\; k \geq k_*,\; k \in \mathcal{K}_* \;\text{ with }\; \Delta_k \leq \delta_*. \tag{3.30}$$

We may also use Theorem 3.1, AM.3 and (3.23) at x_k to obtain that, for some $k_\chi \geq 0$ sufficiently large,

$$\chi_k \leq \tfrac{1}{2}\chi(x_k) \leq \tfrac{1}{4}\chi_* < 0 \tag{3.31}$$

for all $k \in \mathcal{K}_*$ such that $k \geq k_\chi$.

We now temporarily fix s such that $k_s \geq \max[k_*, k_\chi]$ and distinguish two cases. The first is when

$$\Delta_k \leq \delta_* \tag{3.32}$$

for all $k_s \leq k \leq q_s$. Using the definition of q_s, the triangular inequality, the inclusion $x_{k_s} \in \mathcal{V}_{\rm in}$ and (3.26), we observe that

$$\delta_{\rm out} < \|x_{q_s+1} - x_*\| \leq \|x_{q_s+1} - x_{k_s}\| + \|x_{k_s} - x_*\| \leq \|x_{q_s+1} - x_{k_s}\| + \tfrac{1}{2}\delta_{\rm out}. \tag{3.33}$$

Hence, since (3.30) and (3.32) imply that all iterations k_s to q_s must be very successful, we have that

$$\begin{aligned} \tfrac{1}{2}\delta_{\rm out} &< \|x_{q_s+1} - x_{k_s}\| \leq \sum_{\ell=k_s}^{q_s} \|x_{\ell+1} - x_\ell\| \\ &\leq \sum_{\ell=k_s}^{q_s} \Delta_\ell \leq \sum_{\ell=k_s}^{q_s} \frac{\Delta_{q_s}}{\gamma_3^{\ell-k_s}} \leq \frac{\gamma_3}{\gamma_3 - 1}\Delta_{q_s}, \end{aligned} \tag{3.34}$$

where we used (3.33) to derive the first inequality and AA.4 to ensure that the last fraction is well-defined. Gathering (3.34) and (3.29), we obtain that

$$\Delta_{t_s} \geq \frac{(\gamma_3 - 1)\delta_{\rm out}}{2\gamma_3\gamma_4^m} \stackrel{\rm def}{=} \delta_a > 0. \tag{3.35}$$

Using this bound, the inclusion $t_s \in \mathcal{S} \cap \mathcal{T} \cap \mathcal{K}_*$, (2.8) and (3.31), we then conclude that

$$f(x_{t_s}) - f(x_{t_s+1}) \geq \eta_1[m_{t_s}(x_{t_s}) - m_{t_s}(x_{t_s+1})] \geq \tfrac{1}{4}\eta_1\kappa_{\rm wcc}|\chi_*| \min[\tfrac{1}{16}\chi_*^2, \delta_a^2]. \tag{3.36}$$

The second case is when (3.32) fails. In this case, we deduce that the largest trust-region radius for iterations from k_s to q_s is larger than δ_*. Let this maximum radius be attained at iteration v_s ($k_s \leq v_s \leq q_s$). But, since iterations for which $\Delta_k \leq \delta_*$ must be very successful because of (3.30), the mechanism of the algorithm ensures that $\Delta_k \geq \gamma_1\delta_*$ for all iterations between v_s and q_s. In particular, $\Delta_{q_s} \geq \gamma_1\delta_*$, which in turn gives that

$$\Delta_{t_s} \geq \frac{\gamma_1\delta_*}{\gamma_4^m} \stackrel{\rm def}{=} \delta_b > 0 \tag{3.37}$$

because of (3.29). This then yields that

$$f(x_{t_s}) - f(x_{t_s+1}) \geq \eta_1[m_{t_s}(x_{t_s}) - m_{t_s}(x_{t_s+1})] \geq \tfrac{1}{4}\eta_1\kappa_{\rm wcc}|\chi_*| \min[\tfrac{1}{16}\chi_*^2, \delta_b^2], \tag{3.38}$$

where we successively used the inclusion $t_s \in \mathcal{S} \cap \mathcal{T} \cap \mathcal{K}_*$, (2.8), (3.31) and (3.37).

Combining (3.36), (3.38) and the non-increasing nature of the sequence $\{f(x_k)\}$, we thus obtain that

$$f(x_{k_s}) - f(x_{q_s+1}) \geq f(x_{t_s}) - f(x_{t_s+1}) \geq \tfrac{1}{4}\eta_1 \kappa_{\rm wcc} |\chi_*| \min[\tfrac{1}{16}\chi_*^2, \delta_a^2, \delta_b^2] > 0.$$

Summing now this last inequality over the subsequence $\{k_s\}$ for $k_s \geq \max[k_*, k_\chi]$, we obtain that the non-increasing sequence $\{f(x_k)\}$ has to tend to minus infinity, which contradicts our assumption AF.2 that the objective function is bounded below. Thus the sequence of iterates can only leave $\mathcal{V}_{\rm out}$ a finite number of times. Since we have shown that it is infinitely often inside this neighbourhood but cannot remain in it either, we obtain the desired contradiction. Hence our assumption that $\chi_* < 0$ must be false and the proof is complete because of Theorem 3.1. □

4 Application to recursive multigrid trust-region methods

As indicated above, we now wish to apply these results to multilevel recursive trust-region methods of the type investigated in [3]. These methods are applicable to the discretized form of optimization problems arising in infinite dimensional settings. They use the problem formulation at different discretization levels, from the coarsest (level 0) to the finest (level r), where each level i involves n_i variables. An efficient subclass of recursive trust-region methods, the multigrid-type V-cycle variant (whose details are discussed in Section 3 of [3]), can be broadly described as the determination of a starting point for level r, followed by an alternating succession of *smoothing* and *recursive* iterations from level r.

Smoothing iterations at level i ($i > 0$) consists in one or more cycles during which an (exact) quadratic model of the objective function at level i is minimized, within the trust region, along each direction of the coordinate basis in turn. This minimization may be organized (see [3] for details) to ensure that, if $\mu_{i,k}$, the most negative diagonal element of model's Hessian at the current iterate, is negative, then

$$m_{i,k}(x_{i,k}) - m_{i,k}(x_{i,k} + s_{i,k}) \geq \tfrac{1}{2}|\mu_{i,k}|\Delta_{i,k}^2, \tag{4.1}$$

where $x_{i,k}$ is the current iterate at iteration k and level i, $m_{i,k}$ the model for this iteration and $s_{i,k}$ the direction resulting from the smoothing cycle(s) at this iteration. This corresponds, for level r, to (2.8) with

$$\mathcal{D} = \{e_{r,j}\}_{j=1}^{n_r} \stackrel{\rm def}{=} \mathcal{D}_s, \tag{4.2}$$

where we denote by $e_{i,j}$ the jth vector of the coordinate basis at level i.

Recursive iterations at level $i > 0$, on the other hand, use a restriction operator P_i^T to "project" the model of the objective function onto level $i-1$. If this level is the coarsest (level 0), the projected model is minimized exactly within the trust region, yielding the inequality (2.6) at level 0 if $\tau_{0,k} < 0$. If level $i-1$ is not the coarsest ($i > 1$), then the algorithm is recursively applied in the sense that a smoothing iteration is performed at level $i-1$, followed by a recursive iteration at level $i-1$, itself possibly followed by a further smoothing iteration at level $i-1$. This results in an overall step s_{i-1} at level $i-1$, which is then prolongated by interpolation to level i to give a step at level i of the form $s_i = P_i s_{i-1}$. Recursive iterations at level r can then be viewed as iterations where negative curvature is tested (and exploited) in the subspace corresponding to the coarsest variables, $\mathcal{V}_0$, say, as well as along the coordinate vectors at each level i for $i = 1, \ldots, r-1$. Translated in terms of finest level variables, this means that (2.8) holds for recursive iterations if $\chi_k < 0$ and with

$$\mathcal{D} = \{P_r \ldots P_1 v \mid v \in \mathcal{V}_0 \text{ and } \|v\| = 1\} \cup \left[\bigcup_{i=1}^{r-1} \{q_{i,j}\}_{j=1}^{n_i}\right] \stackrel{\text{def}}{=} \mathcal{D}_r, \tag{4.3}$$

where

$$q_{i,j} = \frac{P_r \ldots P_{i+1} e_{i,j}}{\|P_r \ldots P_{i+1} e_{i,j}\|}.$$

Ignoring the calculation of the starting point, one may therefore see the V-cycle recursive trust-region algorithm as a succession of iterations where (2.8) holds for $\chi_k < 0$ either with $\mathcal{D} = \mathcal{D}_s$ or $\mathcal{D} = \mathcal{D}_r$. If we additionally insist on repeating smoothing or recursive iterations until they are successful (as to ensure AA.1), we may then ignore unsuccessful iterations (as above) and deduce that, under the assumptions of Section 3, all limit points of the sequence of iterates are weakly second-order critical with respect to all directions in $\mathcal{D}_s \cup \mathcal{D}_r$.

One might also try to exploit the initialization phase of the algorithm to approach a (weakly) second-order critical point. This initialization is usually performed by applying the same algorithm to (possibly approximately) solve the problems at levels 0 to $r-1$ successively, each time interpolating the solution at level i to obtain the starting value at level $i+1$. In [1], Borzi and Kunisch propose to use a method ensuring convergence to second-order critical points for the first step of this procedure, that is for the computation of the initial solution at the coarsest grid. While this idea is often helpful in practice, it may not be sufficient in general because there is no guarantee that iterates at finer levels will remain in the "neighbourhood" of the solution found at the first stage of initialization. Even if they do, second-order criticality at coarse levels does not imply second-order criticality at higher ones, as is shown by the following simple example.

Consider the problem of finding the functions $x(t)$ and $y(t)$ on the real interval $[0, 1]$ that solve the problem

$$\min_{x,y} \int_0^1 f(x(t), y(t), w(t))\, dt, \tag{4.4}$$

where $w(t) = [\sin(2^k \pi t)]^2 \in [0, 1]$ for some positive integer k and

$$f(x, y, w) = 2w[(x-0.9)^2+y^2-1.21]^2+0.72[(x-1+\cos(\pi w))^2+(y-\sin(\pi w))^2]. \tag{4.5}$$

If we discretize the interval into $n_i \stackrel{\text{def}}{=} 2^i$ equal subintervals of length $h_i \stackrel{\text{def}}{=} 1/2^i$ and approximate the integral by using the trapezoidal rule, the discretized problem takes the form

$$\min_{\substack{x_j, y_j \\ j=0,\ldots,n_i}} \left[\tfrac{1}{2} f(x_0, y_0, w_0) + \sum_{j=1}^{n_i - 1} f(x_j, y_j, w_j) + \tfrac{1}{2} f(x_{n_i}, y_{n_i}, w_{n_i}) \right] h_i \tag{4.6}$$

with $x_j = x(jh_i)$, $y_j = y(jh_i)$ and $w_j = w(jh_i)$. An analysis of (4.5) reveals that f admits a unique minimizer $(x_*(w), y_*(w))$ for each $w \in [0,1]$. This minimizer is the origin for $w = 0$, the point $(2,0)$ for $w = 1$ and a point in the positive orthant for $w \in (0,1)$ (see Figure 4.1). Moreover, f is a convex quadratic for $w = 0$ but admits a saddle point at the origin for $w = 1$, with a direction of negative curvature (-3.2) along the y-axis.

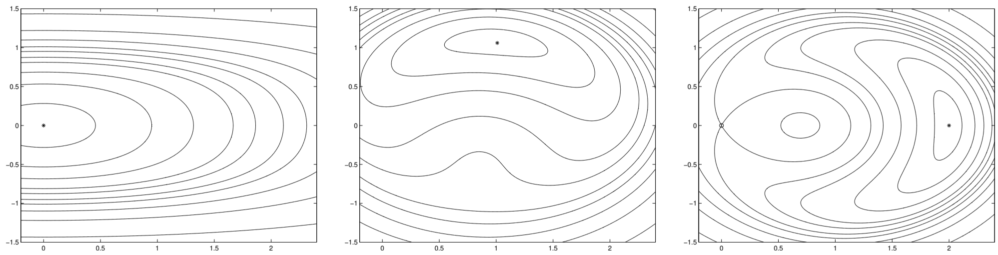

Figure 4.1: The contours of $f(x,y,w)$ for $w = 0$, $w = 0.5$ and $w = 1$ (from left to right). A minimum is indicated by a star and a saddle point by a circle.

Now observe that $w_j = 0$ whenever $i \le k$. Hence the only first-order critical point for the problem is $x_j = y_j = 0$ ($j = 0, \ldots, n_i$) for $i \le k$, and this point is also second-order critical. If we now consider $i = k+1$, we see that $w_j = 1$ for j odd. Hence the unique (second-order critical) solution of the problem is given, for $j = 0, \ldots, n_i$, by

$$x_j = \begin{cases} 0 & \text{for } j \text{ even }, \\ 2 & \text{for } j \text{ odd }, \end{cases} \quad \text{and} \quad y_j = 0. \tag{4.7}$$

On the other hand, interpolating (linearly) the solution of the problem for $i = k$ to a tentative solution for the problem with $i = k+1$ gives $x_j = y_j = 0$ for $j = 0, \ldots, n_i$. This tentative solution is now a first-order critical point, but not a second-order one since the curvature along the y-axis is equal to -3.2. Thus Borzi and Kunisch's technique would typically encounter problems on this example, because nonconvexity occurs at levels that are not the coarsest. On the other hand, the multilevel recursive trust-region method discussed above detects negative curvature along the coordinate vectors at high levels, and our result guaranteeing weakly second-order critical limit points applies. Finally observe that we could have derived the example as arising from a (one dimensional) partial differential equation problem: such a problem may be obtained for instance by defining the function $w(t)$ as the solution of the simple differential problem

$$\frac{1}{2^{2k+1}\pi^2} \frac{d^2 w}{dt^2}(t) + w(t) = [\cos(2^k \pi t)]^2 \quad \text{with} \quad w(0) = w(1) = 0 \tag{4.8}$$

and considering (4.4) and (4.8) as an equality constrained optimization problem in $x(t)$, $y(t)$ and $w(t)$. The extension of the example to more than one dimension is also possible without difficulty. Observe finally that any other reasonable interpolation of the solution from level k to level $k+1$ would also yield the same results.

5 Conclusions and extensions

We have presented a convergence theory for trust-region methods that ensures convergence to weakly second-order critical points. This theory allows for incomplete curvature information, in the sense that only some directions of the space are investigated and this investigation is only carried out for a subset of iterations. Fortunately, the necessary additional assumptions are algorithmically realistic and suggest minor modifications to existing methods. The concepts apply well to multilevel recursive trust-region methods, for which they provide new optimality guarantees. They also provide a framework in which methods requiring the explicit computation of the most negative eigenvalue of the Hessian can be made less computationally expensive.

Many extensions of the ideas discussed in this paper are possible. Firstly, only the Euclidean norm has been considered above, but the extension to iteration dependent norms, and hence iteration dependent trust-region shapes, is not difficult. As in [2], a uniform equivalence assumption is sufficient to directly extend our present results. The assumption that the set $\mathcal{D}$ is closed can be relaxed by introducing an infimum instead of a minimum in the definition of χ_k, provided this can be calculated, and a supremum instead of a maximum in AM.3. A further generalization may be obtained by observing that the second-order conditions (2.8) and (2.10) play no role unless a first-order critical point is approached. As a consequence, they only need to hold for sufficiently small values $\|g_k\|$ for the theory to apply, as is already the case for AM.3.

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