CORRIGENDUM: NONLINEAR PROGRAMMING WITHOUT A PENALTY FUNCTION OR A FILTER

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Report NAXYS-07-2011 3 February 2011

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Abstract

A new method is introduced for solving equality constrained nonlinear optimization problems. This method does not use a penalty function, nor a filter, and yet can be proved to be globally convergent to first-order stationary points. It uses different trust-regions to cope with the nonlinearity of the objective and constraint functions, and allows inexact SQP steps that do not lie exactly in the nullspace of the local Jacobian. Preliminary numerical experiments on CUTEr problems indicate that the method performs well.

Keywords: Nonlinear optimization, equality constraints, numerical algorithms, global convergence.

Context

This paper presents a correction to the results obtained by Gould and Toint (2010), in which an error was unfortunately discovered. The problem is in the proof of Lemma 3.10 of this reference, where it is claimed that Lemma 6.5.3 of Conn, Gould and Toint (2000) can be invoked to deduce that $\rho_k^c \geq \eta_2$, where ρ_k^c is a specific ratio of achieved to predicted reduction is constraint violation and η_2 is a constant in $(0,1)$. As it turns out, the reasonning is only correct if the ratio $\|s_k\|/\|s_k^R\|$ is bounded above, where s_k is the step at iteration k and s_k^R is its projection onto the range of the transposed Jacobian J_k^T.

Handling the case where this ratio is unbounded above turned out to be surprisingly complex. In particular, this required considering separately the cases where the tangential component of the step at iteration k is large or small with respect to its normal component, where the meaning of "large" and "small" has to be defined very specifically. The convergence proof taking this distinction into account is therefore significantly more involved than the proof of Gould and Toint (2010), and cannot be discussed in the form of a few corrections in the original text. It is the purpose of the present paper to propose a corrected version of Gould and Toint (2010), where other minor improvements and updates have also been introduced, including fixing a problematic case where it was possible to skip the normal step computation although the current iterate was close to the current infeasibility limit.

1 Introduction

We consider the numerical solution of the equality constrained nonlinear optimization problem

$$\left\{ \begin{array}{ll} \min\limits_{x} & f(x) \\ & c(x) = 0, \end{array} \right. \tag{1.1}$$

where we assume that $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable and that f is bounded below on the feasible domain.

The present paper introduces a new method for the solution of (1.1), which belongs to the class of trust-region methods for constrained optimization, in the spirit of Omojokun (1989) in a Ph.D. thesis supervised by R. Byrd, and later developed by several authors, including Biegler, Nocedal and Schmid (1995), El-Alem (1995, 1999), Byrd, Gilbert and Nocedal (2000*a*), Byrd, Hribar and Nocedal (2000*b*), Liu and Yuan (2000) and Lalee, Nocedal and Plantenga (1998) (also see Chapter 15 of Conn et al., 2000).

The algorithm presented here has four main features. The first is that it attempts to consider the objective function and the constraints as independently as possible by using different models and trust regions for f and c. As is common to the methods cited, the steps are computed as a combination of normal and tangential components, the first aiming to reduce the constraint violation, and the second at reducing the objective function while retaining the improvement in violation by remaining in the plane tangent to the constraints, but only approximately so. This framework can thus be viewed as a sequential quadratic programming technique that allows for inexact tangential steps, which is the second main characteristic of our proposal (shared with Heinkenschloss and Vicente, 2001, Byrd, Curtis and Nocedal, 2008 and 2010, and Curtis, Schenk and Wächter, 2010). The third distinctive feature is that the algorithm is not compelled to compute both normal and tangential steps at every iteration, rather only to compute whichever is/are likely to improve feasibility and optimality significantly. Thus if an iterate is almost feasible, there is little point in trying to further improve feasibility while the objective value is far from optimal. The final central feature is that the algorithm does not use any merit function (penalty, or otherwise), thereby avoiding the practical problems associated with the setting of the merit function parameters, but nor does it use the filter idea first proposed by Fletcher and Leyffer (2002). Instead, the convergence is driven by the *trust funnel*, a progressively decreasing limit on the permitted infeasibility of the successive iterates.

It is, in that sense and albeit very indirectly, reminiscent of the "flexible tolerance method" by Himmelblau (1972), but also of the "tolerance tube method" by Zoppke-Donaldson (1995). It also has similarities with the SQP methods by Yamashita and Yabe (2004), Ulbrich and Ulbrich (2003) and Bielschowsky and Gomes (2008). All these methods use the idea of progressively reducing constraint violation to avoid using a penalty parameter. The four more modern algorithms are of the trust-region type, but differ significantly from our proposal. The first major difference is that they all require the tangential component of the step to lie exactly in the Jacobian's nullspace: they are thus "exact" rather than "inexact" SQP methods. The second is that they all use a single trust region to account simultaneously for constraint violation and objective function improvement. The third is that both limit constraint violation *a posteriori*, once the true nonlinear constraints have been evaluated, rather than attempting to limit its predicted value *a priori*. The "tolerance tube" method resorts to standard second-order correction steps when the iterates become too infeasible. No convergence seems to be available for the method, although the numerical results appear satisfactory. At variance, the method by Yamashita and Yabe (2004), itself motivated by an earlier report by Yamashita (1979), is provably globally convergent to first-order critical points and involves a combination of linesearch and trust-regions. The normal step is computed by solving a quadratic program involving the Hessian of the problem's Lagrangian, while the tangential step requires the solution of one linear and two quadratic programs. The method by Ulbrich and Ulbrich (2003) computes a composite SQP step and accepts the resulting trial iterate on the basis of non-monotone tests which require both a sufficient reduction of infeasibility and an improvement in optimality. Global and fast asymptotic convergence (without the Maratos effect) is proved for the resulting algorithm. Finally, the algorithm by Bielschowsky and Gomes (2008) is also provably globally convergent to first-order critical points. It however involves a "restoration" phase (whose convergence is assumed) to achieve acceptable constraint violation in which the size of normal component of the step is restricted to be a fraction of the current infeasibility limit. This limit is updated using the gradient of the Lagrangian function, and the allowable fraction is itself computed from the norm of exact projection of the objective function gradient onto the nullspace of the constraints' Jacobian.

The paper is organized as follows. Section 2 introduces the new algorithm, whose convergence theory is presented in Section 3. Conclusions and perspectives are finally outlined in Section 4.

2 A trust-funnel algorithm

2.1 The normal step

Let us measure, for any x, the constraint violation at x by

$$\theta(x) \stackrel{\text{def}}{=} \tfrac{1}{2}\|c(x)\|^2 \tag{2.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. Now consider iteration k, starting from the iterate x_k, for which we assume we know a bound $\theta_k^{\max}$ such that $\frac{1}{2}\|c(x_k)\|^2 < \theta_k^{\max}$.

Firstly, a *normal step* n_k[(1)] is computed if the constraint violation is significant (in a sense to be defined shortly). This is achieved by reducing the Gauss-Newton model

$$\tfrac{1}{2}\|c_k + J_k n\|^2 \tag{2.2}$$

of $\theta(x_k+n_k)$—here we write $c_k \stackrel{\text{def}}{=} c(x_k)$ and $J_k \stackrel{\text{def}}{=} J(x_k)$ is the Jacobian of c at x_k—while requiring that n_k remains in the "normal trust region", i.e.,

$$n_k \in \mathcal{N}_k \stackrel{\text{def}}{=} \{v \in \mathbb{R}^n \mid \|v\| \le \Delta_k^c\}, \tag{2.3}$$

for some radius $\Delta_k^c > 0$. More formally, this Gauss-Newton-type step is computed by choosing n_k so that (2.2) is reduced sufficiently within $\mathcal{N}_k$ in the sense that

$$\delta_k^{c,n} \stackrel{\text{def}}{=} \tfrac{1}{2}\|c_k\|^2 - \tfrac{1}{2}\|c_k + J_k n_k\|^2 \ge \kappa_{\text{nC}}\|J_k^T c_k\| \min\left[\frac{\|J_k^T c_k\|}{1+\|W_k\|}, \Delta_k^c\right] \ge 0, \tag{2.4}$$

where $W_k = J_k^T J_k$ is the symmetric Gauss-Newton approximation of the Hessian of θ at x_k and $\kappa_{\text{nC}} \in (0, \frac{1}{2}]$. Condition (2.4) is nothing but the familiar Cauchy condition for problem approximately minimizing (2.2) within the region $\mathcal{N}_k$.

In addition to (2.4), we also require the normal step to be "normal", in that it mostly lies in the space spanned by the columns of the matrix J_k^T by imposing that

$$\|n_k\| \le \kappa_{\text{n}}\|c_k\| \tag{2.5}$$

for some $\kappa_{\text{n}} \ge 1$. These conditions on the normal step are very reasonable in practice, as it is known that they hold, for instance, if n_k is computed by applying one or more steps of a truncated conjugate-gradient method (see Steihaug, 1983, and Toint, 1981) to the minimization of the square of the linearized infeasibility. Other Krylov-space based techniques, such as LSQR (see Paige and Saunders, 1982) or LSTR (see Cartis, Gould and Toint, 2009) also guarantee that these conditions hold, as is the case if the model $\frac{1}{2}\|c_k + J_k n\|^2$ is minimized exactly in $\mathcal{N}_k$. Note that the conditions (2.3), (2.4) and (2.5) allow us to choose a null normal step ($n_k = 0$) if x_k is feasible.

2.2 The tangential step

Having computed the normal step, we next consider if some improvement is possible on the objective function, while not jeopardizing the infeasibility reduction we have just obtained. Because of this latter constraint, it makes sense to remain in $\mathcal{N}_k$, the region where we believe that our model of constraint violation can be trusted, but we also need

[(1)] Not to be confused with n, the number of variables.

to trust the model of the objective function given, as is traditional in sequential quadratic programming (see Section 15.2 of Conn et al., 2000), by

$$m_k(x_k + n_k + t) = f_k + \langle g_k, n_k \rangle + \tfrac{1}{2}\langle n_k, H_k n_k \rangle + \langle g_k^{\mathrm{N}}, t \rangle + \tfrac{1}{2}\langle t, G_k t \rangle \tag{2.6}$$

where

$$g_k^{\mathrm{N}} \stackrel{\text{def}}{=} g_k + G_k n_k, \tag{2.7}$$

where $f_k = f(x_k)$, $g_k = \nabla f(x_k)$ and where G_k is a symmetric approximation of the Hessian of the Lagrangian $\ell(x,y) = f(x) + \langle y, c(x) \rangle$ given by

$$G_k \stackrel{\text{def}}{=} H_k + \sum_{i=1}^{m} [\hat{y}_k]_i C_{ik}. \tag{2.8}$$

In this last definition, H_k is a bounded symmetric approximation of $\nabla^2 f(x_k)$, the matrices C_{ik} are bounded symmetric approximations of the constraints' Hessians $\nabla_{xx} c_i(x_k)$ and the vector $\hat{y}_k$ may be viewed as a bounded approximation of the local Lagrange multipliers, in the sense that we require that

$$\|\hat{y}_k\| \leq \kappa_{\mathrm{y}} \tag{2.9}$$

for some $\kappa_{\mathrm{y}} > 0$. We assume that (2.6) can be trusted as a representation of $f(x_k + n_k + t)$ provided the complete step $s = n_k + t$ belongs to

$$\mathcal{T}_k \stackrel{\text{def}}{=} \{ s \in \mathbb{R}^n \mid \|s\| \leq \Delta_k^f \}, \tag{2.10}$$

for some radius Δ_k^f. Thus our attempts to reduce (2.6) should be restricted to the intersection of $\mathcal{N}_k$ and $\mathcal{T}_k$, which imposes that the *tangential step* t_k results in a complete step $s_k = n_k + t_k$ that satisfies the inclusion

$$s_k \in \mathcal{B}_k \stackrel{\text{def}}{=} \mathcal{N}_k \cap \mathcal{T}_k \stackrel{\text{def}}{=} \{ s \in \mathbb{R}^n \mid \|s\| \leq \Delta_k \}, \tag{2.11}$$

where the radius Δ_k of $\mathcal{B}_k$ is thus given by

$$\Delta_k = \min[\Delta_k^c, \Delta_k^f]. \tag{2.12}$$

As a consequence, it makes sense to ask n_k to belong to $\mathcal{B}_k$ before attempting the computation of t_k, which we formalize by requiring that

$$\|n_k\| \leq \kappa_{\mathrm{B}} \Delta_k, \tag{2.13}$$

for some $\kappa_{\mathrm{B}} \in (0,1)$. We note here that using two different trust-region radii can be considered as unusual, but is not unique. For instance, the SLIQUE algorithm described by Byrd, Gould, Nocedal and Waltz (2004) also uses different radii, but for different models of the same function, rather than for two different functions.

We still have to specify what we mean by "reducing (2.6)", as we are essentially interested in the reduction in the hyperplane tangent to the constraints. In order to compute an approximate projected gradient at $x_k + n_k$, we first compute a new local estimate of the Lagrange multipliers y_k such that

$$\|y_k + [J_k^T]^I g_k^{\mathrm{N}}\| \leq \omega_y(\|c_k\|) \tag{2.14}$$

for some monotonic bounding function[(2)] ω_y, the superscript ^{I} denoting the Moore-Penrose generalized inverse, and such that

$$\|r_k\| \leq \kappa_{\mathrm{nr}} \|g_k^{\mathrm{N}}\| \tag{2.15}$$

[(2)]Here and later in this paper, a *bounding function* ω is defined to be a continuous function from $\mathbb{R}_+$ into $\mathbb{R}$ with the property that $\omega(t)$ converges to zero as t tends to zero.

for some $\kappa_{\rm nr} > 0$, and

$$\langle g_k^{\rm N}, r_k \rangle \geq 0, \tag{2.16}$$

where

$$r_k \stackrel{\rm def}{=} g_k^{\rm N} + J_k^T y_k \tag{2.17}$$

is an approximate projected gradient of the model m_k at $x_k + n_k$. Conditions (2.14)–(2.16) are reasonable since they are obviously satisfied by choosing y_k to be a solution of the least-squares problem

$$\min_y \tfrac{1}{2} \| g_k^{\rm N} + J_k^T y \|^2, \tag{2.18}$$

and thus, by continuity, by sufficiently good approximations of this solution. In practice, one can compute such an approximation by applying a Krylov space iterative method starting from $y = 0$. If the solution of (2.18) is accurate, r_k is the orthogonal projection of $g_k^{\rm N}$ onto the nullspace of J_k, which then motivates that we require the tangent step to produce a reduction in the model m_k which is at least a fraction of that achieved by solving the modified Cauchy point subproblem

$$\min_{\substack{\tau > 0 \\ x_k + n_k - \tau r_k \in \mathcal{B}_k}} m_k(x_k + n_k - \tau r_k), \tag{2.19}$$

where we have assumed that $\|r_k\| > 0$. We know from Section 8.1.5 of Conn et al. (2000) that this procedure ensures, for some $\kappa_{\rm tC1} \in (0, 1]$, the modified Cauchy condition

$$\delta_k^{f,t} \stackrel{\rm def}{=} m_k(x_k + n_k) - m_k(x_k + n_k + t_k) \geq \kappa_{\rm tC1} \pi_k \min \left[\frac{\pi_k}{1 + \|G_k\|}, \tau_k \|r_k\| \right] > 0 \tag{2.20}$$

on the decrease of the objective function model within $\mathcal{B}_k$, where we have set

$$\pi_k \stackrel{\rm def}{=} \frac{\langle g_k^{\rm N}, r_k \rangle}{\|r_k\|} \geq 0 \tag{2.21}$$

(by convention, we define $\pi_k = 0$ whenever $r_k = 0$), and where τ_k is the maximal step length along $-r_k$ from $x_k + n_k$ which remains in the trust-region $\mathcal{B}_k$. But we have that

$$\tau_k \|r_k\| \geq (1 - \kappa_{\rm B}) \Delta_k$$

by construction and thus the modified Cauchy condition (2.20) may now be rewritten as

$$\delta_k^{f,t} \stackrel{\rm def}{=} m_k(x_k + n_k) - m_k(x_k + n_k + t_k) \geq \kappa_{\rm tC} \pi_k \min \left[\frac{\pi_k}{1 + \|G_k\|}, \Delta_k \right] \tag{2.22}$$

with $\kappa_{\rm tC} \stackrel{\rm def}{=} \kappa_{\rm tC1}(1 - \kappa_{\rm B}) \in (0, 1)$. We see from (2.22) that π_k may be considered as an optimality measure in the sense that it measures how much decrease could be obtained locally along the negative of the approximate projected gradient r_k. This role as an optimality measure is confirmed in Lemma 3.2 below.

Our last requirement on the tangential step t_k is to ensure that it does not completely "undo" the improvement in linearized feasibility obtained from the normal step without good reason. We consider two possible situations. The first is when the predicted decrease in the objective function is substantial compared to its possible deterioration along the normal step and the step is not too large compared to the maximal allowable infeasibility, i.e. when both

$$\delta_k^{f,t} \geq -\bar{\kappa}_\delta \, \delta_k^{f,n} \tag{2.23}$$

and

$$\|s_k\| \leq \kappa_\Delta \sqrt{\theta_k^{\max}}, \tag{2.24}$$

for some $\bar{\kappa}_\delta > 1$ and some $\kappa_\Delta > 0$, where

$$\delta_k^{f,n} \stackrel{\rm def}{=} m_k(x_k) - m_k(x_k + n_k).$$

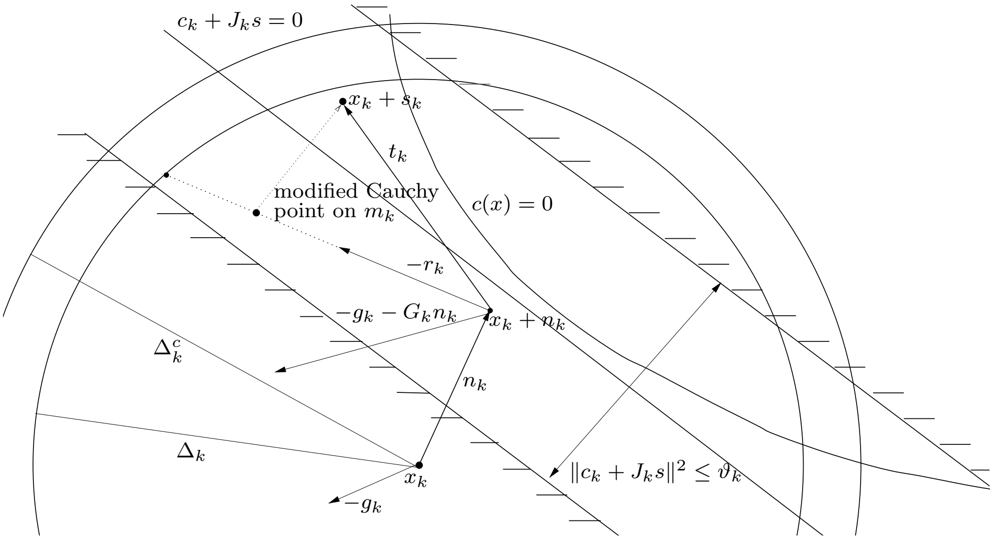

Figure 2.1: The components of a step s_k satisfying (2.26) in the case where $\Delta_k = \Delta_k^f$.

When (2.23) and (2.24) hold, we allow more freedom in the linearized feasibility and merely require that

$$\tfrac{1}{2}\|c_k + J_k(n_k + t_k)\|^2 \leq \kappa_{\rm tt}\theta_k^{\max} \tag{2.25}$$

for some $\kappa_{\rm tt} \in (0,1)$. If, on the other hand, (2.23) or (2.24) fails, meaning that we cannot hope to trade some decrease in linearized feasibility for a large improvement in objective function value over a reasonable step, then we require that the tangential step satisfies

$$\|c_k + J_k(n_k + t_k)\|^2 \leq \kappa_{\rm tg}\|c_k\|^2 + (1 - \kappa_{\rm tg})\|c_k + J_k n_k\|^2 \stackrel{\rm def}{=} \vartheta_k, \tag{2.26}$$

for some $\kappa_{\rm tg} \in (0,1)$. Note that this inequality is already satisfied at the end of the normal step since $\|c_k + J_k n_k\| \leq \|c_k\|$ and thus already provides a relaxation of the (linearized) feasibility requirement at $x_k + n_k$. Note also that $\vartheta_k \leq \|c_k\|^2$, an observation which we will use below. Figure 2.1 illustrate the geometry of the various quantities involved in the construction of a step s_k satisfying (2.26).

2.3 Which steps to compute and retain

We now observe that a tangential step does not make too much sense if $r_k = 0$, and we do not compute any in this case. By convention we then choose to define $\pi_k = 0$ and $t_k = 0$. The situation is similar if π_k is small compared to the current infeasibility. Given a monotonic bounding function ω_t, we thus decide that if

$$\pi_k > \omega_t(\|c_k\|), \tag{2.27}$$

fails, then the current iterate is still too far from feasibility to worry about optimality, and we again skip the tangential step computation by setting $t_k = 0$.

In the same spirit, we have imposed above the current violation to be "significant" as a condition to compute the normal step n_k, but didn't specify what we formally meant, because our optimality measure π_k was not defined at that point. We now complete our description by requiring that the computation of the normal step only when $k = 0$ or

$$\|c_k\| > \omega_n(\pi_{k-1}) \;\text{ or }\; \theta_k > \kappa_{\theta\theta}\theta_k^{\max} \tag{2.28}$$

where ω_n is some bounding function, $\kappa_{\theta\theta} \in (0,1)$ is a constant and $\theta_k \stackrel{\text{def}}{=} \theta(x_k)$. If (2.28) fails, we remain free to compute a normal step, but we may also skip it. In this latter case, we simply set $n_k = 0$. For technical reasons which will become clear below, we impose the additional conditions that

$$\omega_n(t) = 0 \Longleftrightarrow t = 0 \quad \text{and} \quad \omega_t(\omega_n(t)) \leq \kappa_\omega t \tag{2.29}$$

for all $t \geq 0$ and for some $\kappa_\omega \in (0,1)$.

While (2.27) and (2.28) together provide considerable flexibility in our algorithm in that a normal or tangential step is only computed when relevant, our setting also produces the possibility that both these conditions fail. In this case, we have that $s_k = n_k + t_k$ is identically zero, and the sole computation in the iteration is that of the new Lagrange multiplier y_k; we will actually show that such behaviour cannot persist unless x_k is optimal.

Finally, we may evaluate the usefulness of the tangential step t_k after (or during) its computation, in the sense that we would like a relatively large tangential step to cause a clear decrease in the model (2.6) of the objective function. We therefore check whether the conditions

$$\|t_k\| > \kappa_{\mathcal{CS}} \|n_k\| \tag{2.30}$$

and

$$\delta_k^f \stackrel{\text{def}}{=} \delta_k^{f,t} + \delta_k^{f,n} \geq \kappa_\delta \delta_k^{f,t} \tag{2.31}$$

are satisfied for some $\kappa_{\mathcal{CS}} > 1$ and for $\kappa_\delta = 1 - 1/\bar{\kappa}_\delta \in (0,1)$. The latter inequality is equivalent to (2.23) and indicates that the *predicted* improvement in the objective function obtained in the tangential step is not negligible compared to the *predicted* change in f resulting from the normal step. If (2.30) holds but (2.31) fails, the tangential step is not useful in the sense discussed at the beginning of this pargagraph, and we choose to ignore it by resetting $t_k = 0$.

2.4 Iterations types

Once we have computed the step s_k and the trial point

$$x_k^+ \stackrel{\text{def}}{=} x_k + s_k \tag{2.32}$$

completely, we are left with the task of accepting or rejecting it. Our proposal is based on the distinction between *y-iterations*, *f-iterations* and *c-iterations*, in the spirit of Fletcher and Leyffer (2002), Fletcher, Leyffer and Toint (2002*b*) or Fletcher, Gould, Leyffer, Toint and Wächter (2002*a*). If $n_k = t_k = 0$, iteration k is said to be a y-iteration because the only computation potentially performed is that of a new vector of Lagrange multiplier estimates. We will say that iteration k is an f-iteration if $t_k \neq 0$, (2.31) holds, and

$$\theta(x_k^+) \leq \theta_k^{\max}. \tag{2.33}$$

Condition (2.33) ensures that the step keeps feasibility within reasonable bounds. Thus the iteration's expected major achievement is, in this case, a decrease in the value of the objective function f, hence its name. If $s_k \neq 0$ and either i) condition (2.31) fails; ii) condition (2.33) fails; or iii) $t_k = 0$ because either (2.13) fails, (2.27) fails, or an initial nonzero tangential step is computed and rejected because it satisfies (2.30) but not (2.31), then iteration k is said to be a c-iteration. If (2.31) fails, then the expected major achievement (or failure) of iteration k is, *a contrario*, to improve feasibility, which is also the case when the step only contains its normal component.

The main idea behind the technique we propose for accepting the trial point is to measure whether the major expected achievement of the iteration has been realized.

- If iteration k is a y-iteration, we do not have any other choice than to restart with $x_{k+1} = x_k$ using the new multipliers. We then define

$$\Delta_{k+1}^f = \Delta_k^f \quad \text{and} \quad \Delta_{k+1}^c = \Delta_k^c \tag{2.34}$$

  and keep the current value of the maximal infeasibility $\theta_{k+1}^{\max} = \theta_k^{\max}$.

- If iteration k is an f-iteration, we accept the trial point if the achieved objective function reduction is comparable to its predicted value along the step s_k. More formally, the trial point is accepted (i.e., $x_{k+1} = x_k^+$) if

$$\rho_k^f \stackrel{\text{def}}{=} \frac{f(x_k) - f(x_k^+)}{\delta_k^f} \geq \eta_1, \tag{2.35}$$

and rejected (i.e., $x_{k+1} = x_k$) otherwise. The radius of $\mathcal{T}_k$ is then updated by

$$\Delta_{k+1}^f \in \begin{cases} [\Delta_k^f, \infty) & \text{if } \rho_k^f \geq \eta_2, \\ [\gamma_2 \Delta_k^f, \Delta_k^f] & \text{if } \rho_k^f \in [\eta_1, \eta_2), \\ [\gamma_1 \Delta_k^f, \gamma_2 \Delta_k^f] & \text{if } \rho_k^f < \eta_1, \end{cases} \tag{2.36}$$

where the constants η_2, γ_1, and γ_2 are given and satisfy the conditions $0 < \eta_1 \leq \eta_2 < 1$ and $0 < \gamma_1 \leq \gamma_2 < 1$, as is usual for trust-region methods. The radius of $\mathcal{N}_k$ is possibly increased if the iteration is successful in the sense that

$$\Delta_{k+1}^c \geq \max\left[\kappa_{\Delta\text{cc}} \|J_{k+1}^T c_{k+1}\|, \Delta_k^c\right] \text{ if } \rho_k^f \geq \eta_1 \tag{2.37}$$

or

$$\Delta_{k+1}^c = \Delta_k^c \text{ if } \rho_k^f < \eta_1, \tag{2.38}$$

for some constant $\kappa_{\Delta\text{cc}} \in (0,1)$. The value of the maximal infeasibility measure is also left unchanged, that is $\theta_{k+1}^{\max} = \theta_k^{\max}$. Note that $\delta_k^f > 0$ (because of (2.22) and (2.31)) unless x_k is first-order critical, and hence that condition (2.35) is well-defined.

- If iteration k is a c-iteration, we accept the trial point if the improvement in feasibility is comparable to its predicted value

$$\delta_k^c \stackrel{\text{def}}{=} \tfrac{1}{2}\|c_k\|^2 - \tfrac{1}{2}\|c_k + J_k s_k\|^2,$$

and the latter is itself comparable to its predicted decrease along the normal step, that is

$$n_k \neq 0, \quad \delta_k^c \geq \kappa_{\text{cn}} \delta_k^{c,n} \quad \text{and} \quad \rho_k^c \stackrel{\text{def}}{=} \frac{\theta(x_k) - \theta(x_k^+)}{\delta_k^c} \geq \eta_1 \tag{2.39}$$

for some $\kappa_{\text{cn}} \in (0, 1 - \kappa_{\text{tg}}]$. If (2.39) fails, the trial point is rejected. The radius of $\mathcal{N}_k$ is then updated by

$$\Delta_{k+1}^c \begin{cases} \in \left[\max\left[\kappa_{\Delta\text{cc}} \|J_{k+1}^T c_{k+1}\|, \Delta_k^c\right], \infty\right) & \text{if } \rho_k^c \geq \eta_2 & \text{and } \delta_k^c \geq \kappa_{\text{cn}} \delta_k^{c,n}, \\ = \max\left[\kappa_{\Delta\text{cc}} \|J_{k+1}^T c_{k+1}\|, \Delta_k^c\right] & \text{if } \rho_k^c \in [\eta_1, \eta_2) & \text{and } \delta_k^c \geq \kappa_{\text{cn}} \delta_k^{c,n}, \\ \in [\gamma_1 \Delta_k^c, \gamma_2 \Delta_k^c] & \text{if } \rho_k^c < \eta_1 & \text{or } \delta_k^c < \kappa_{\text{cn}} \delta_k^{c,n}. \end{cases} \tag{2.40}$$

and that of $\mathcal{T}_k$ is unchanged: $\Delta_{k+1}^f = \Delta_k^f$. We also update the value of the maximal infeasibility by

$$\theta_{k+1}^{\max} = \begin{cases} \max\left[\kappa_{\text{tx1}} \theta_k^{\max}, \theta(x_k^+) + \kappa_{\text{tx2}}(\theta(x_k) - \theta(x_k^+))\right] & \text{if (2.39) hold,} \\ \theta_k^{\max} & \text{otherwise,} \end{cases} \tag{2.41}$$

for some $\kappa_{\text{tx1}} \in (0,1)$ and $\kappa_{\text{tx2}} \in (0,1)$.

We now describe why the last condition in (2.39) is well-defined. Firstly, we only check the third condition *after* the first two conditions have been verified. Assuming that $n_k \neq 0$, the Cauchy condition (2.4) and $c(x_k) \neq 0$ ensure that $\delta_k^{c,n} > 0$ provided $J_k^T c_k \neq 0$. Thus the third condition is well defined, unless $c(x_k) \neq 0$ and $J(x_k)^T c_k = 0$. Such a point x_k is called an infeasible stationary point of θ and is an undesirable situation on which we comment in Section 3. If such a point is encountered, the algorithm is terminated.

Algorithm 2.1: Trust-Funnel Algorithm

Step 0: Initialization. An initial point x_0, an initial vector of multipliers y_{-1} and positive initial trust-region radii Δ_0^f and Δ_0^c are given. Define $\theta_0^{\max} = \max[\kappa_{\rm ca}, \kappa_{\rm cr}\theta(x_0)]$ for some constants $\kappa_{\rm ca} > 0$ and $\kappa_{\rm cr} > 1$. Set $k = 0$.

Step 1: Termination at an infeasible point : If $J(x_k)^T c_k = 0$ and $c(x_k) \neq 0$, terminate the algorithm.

Step 2: Normal step. Possibly compute a normal step n_k that sufficiently reduces the linearized infeasibility (in the sense that (2.4) holds), under the constraint that (2.3) and (2.5) also hold. This computation must be performed if $k = 0$ or if (2.28) holds when $k > 0$.

If n_k has not been computed, set $n_k = 0$.

Step 3: Tangential step. If (2.13) holds, then

Step 3.1: select a vector $\hat{y}_k$ satisfying (2.9) and define G_k by (2.8);

Step 3.2: compute y_k and r_k satisfying (2.14)–(2.17);

Step 3.3: if (2.27) holds, compute a tangential step t_k that sufficiently reduces the model (2.6) (in the sense that (2.22) holds), preserves linearized feasibility enough to ensure either all of (2.23)–(2.25) or (2.26), and such that the complete step $s_k = n_k + t_k$ satisfies (2.11).

If (2.13) fails, set $y_k = 0$. In this case, or if (2.27) fails, or if (2.30) holds but (2.31) fails, set $t_k = 0$ and $s_k = n_k$. In all cases, define $x_k^+ = x_k + s_k$.

Step 4: Conclude a y-iteration. If $n_k = t_k = 0$, then

Step 4.1: accept $x_k^+ = x_k$;

Step 4.2: define $\Delta_{k+1}^f = \Delta_k^f$ and $\Delta_{k+1}^c = \Delta_k^c$;

Step 4.3: set $\theta_{k+1}^{\max} = \theta_k^{\max}$.

Step 5: Conclude an f-iteration. If $t_k \neq 0$ and (2.31) and (2.33) hold,

Step 5.1: accept x_k^+ if (2.35) holds;

Step 5.2: update Δ_k^f according to (2.36) and Δ_k^c according to (2.37)–(2.38);

Step 5.3: set $\theta_{k+1}^{\max} = \theta_k^{\max}$.

Step 6: Conclude a c-iteration. If either $n_k \neq 0$ and $t_k = 0$, or either one of (2.31) or (2.33) fails,

Step 6.1: accept x_k^+ if (2.39) hold;

Step 6.2: update Δ_k^c according to (2.40);

Step 6.3: update the maximal infeasibility $\theta_k^{\max}$ using (2.41).

Step 7: Prepare for the next iteration. If x_k^+ has been accepted, set $x_{k+1} = x_k^+$, else set $x_{k+1} = x_k$. Increment k by one and go to Step 1.

2.5 The trust-funnel algorithm

We are now ready to state our complete algorithm, Algorithm 2.1 on the previous page.

We now comment on Algorithm 2.1. If either (2.35) or (2.39) holds, iteration k is called *successful*. It is said to be *very successful* if, additionally, either $\rho_k^f \geq \eta_2$ or $\rho_k^c \geq \eta_2$, in which case none of the trust-region radii is decreased. We also define the following useful index sets:

$$\mathcal{S} \stackrel{\rm def}{=} \{k \mid x_{k+1} = x_k^+\}, \tag{2.42}$$

the set of successful iterations,

$$\mathcal{Y} \stackrel{\rm def}{=} \{k \mid s_k = 0\}, \quad \mathcal{F} \stackrel{\rm def}{=} \{k \mid t_k \neq 0 \text{ and (2.31) and (2.33) hold}\} \quad \text{and} \quad \mathcal{C} \stackrel{\rm def}{=} \mathbb{N} \backslash (\mathcal{Y} \cup \mathcal{F}),$$

the sets of y-, f- and c-iterations. We further divide this last set into

$$\mathcal{C}_w = \mathcal{C} \cap \{k \mid t_k \neq 0 \text{ and (2.23)–(2.25) hold}\} \quad \text{and} \quad \mathcal{C}_t = \mathcal{C} \setminus \mathcal{C}_w. \tag{2.43}$$

Note that (2.26) must hold for $k \in \mathcal{C}_t$. We finally define

$$\mathcal{A} \stackrel{\rm def}{=} \{k \mid n_k \text{ is computed to satisfy (2.4)}\}.$$

The mechanism of the algorithm ensures that $n_k = 0$ whenever $k \notin \mathcal{A}$, but a null n_k can also happen for $k \in \mathcal{A}$ (if x_k is feasible or is an infeasible stationary point).

We conclude this section by stating a few basic properties of Algorithm 2.1. We first verify that our algorithm is well-defined by deducing a useful "Cauchy-like" condition on the predicted reduction in the infeasibility measure $\theta(x)$ (whose gradient is $J(x)^T c(x)$) over each complete iteration in $\mathcal{A} \cap \mathcal{C}_t$.

Lemma 2.1 *For all $k \in \mathcal{A} \cap \mathcal{C}_t$, the first inequality of (2.39) holds and*

$$\delta_k^c \geq \kappa_{\rm nC2} \|J_k^T c_k\| \min\left[\frac{\|J_k^T c_k\|}{1 + \|W_k\|}, \Delta_k^c\right] \geq 0, \tag{2.44}$$

for $\kappa_{\rm nC2} = (1 - \kappa_{\rm tg})\kappa_{\rm nC}$.

Proof. We first note that our assumption on k implies that (2.26) holds. In this case, we easily verify that

$$\begin{aligned} 2\delta_k^c &= \|c_k\|^2 - \|c_k + J_k s_k\|^2 \\ &\geq \|c_k\|^2 - \kappa_{\rm tg}\|c_k\|^2 - (1 - \kappa_{\rm tg})\|c_k + J_k n_k\|^2 \\ &= (1 - \kappa_{\rm tg}) \left[\|c_k\|^2 - \|c_k + J_k n_k\|^2\right] \end{aligned} \tag{2.45}$$

where we have used (2.26). This and the definition of $\delta_k^{c,n}$ in (2.4) give the first conclusion of the lemma because $\kappa_{\rm cn} \leq 1 - \kappa_{\rm tg}$ by definition. We may now use (2.45) and (2.4) to deduce that

$$\delta_k^c \geq (1 - \kappa_{\rm tg})\kappa_{\rm nC} \|J_k^T c_k\| \min\left[\frac{\|J_k^T c_k\|}{1 + \|W_k\|}, \Delta_k^c\right],$$

and inequality (2.44) then follows. □

We now state an important direct consequence of the definition of our algorithm.

Lemma 2.2 *The sequence $\{\theta_k^{\max}\}$ is non-increasing and the inequality*

$$0 \leq \theta(x_j) \leq \theta_k^{\max} \tag{2.46}$$

holds for all $j \geq k$.

Proof. This results from the initial definition of $\theta_0^{\max}$ in Step 0, the inequality (2.33) (which holds at f-iterations), the fact that $\theta_k^{\max}$ is only updated by formula (2.41) at successful c-iterations, at which Lemma 2.1 ensures that $\delta_k^c > 0$. □

Note that this lemma implies that

$$x_k \in \mathcal{L} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \theta(x) \leq \theta_0^{\max}\}$$

for all $k \geq 0$.

The monotonicity of the sequence $\{\theta_k^{\max}\}$ is what drives the algorithm towards feasibility and, ultimately, to optimality: the iterates can be thought as flowing towards a critical point through a funnel centered on the feasible set. Hence the algorithm's name. We now show a simple useful property of y-iterations.

Lemma 2.3 *For all $k \in \mathcal{Y}$ such that x_k is not an infeasible stationary point,*

$$\pi_k \leq \kappa_\omega \pi_{k-1}.$$

Proof. First suppose that (2.28) is satisfied. This implies that $c_k \neq 0$ and, therefore, $J_k^T c_k \neq 0$ since x_k is not an infeasible stationary point by assumption. Since this condition ensures that a normal step will be computed, we can conclude from (2.4) that $n_k \neq 0$, which is a contradiction. Thus (2.28) must fail. Next, assume that (2.27) is satisfied. Since $n_k = 0$ by assumption, we know that (2.13) holds and thus a tangential step is computed. Since $\pi_k > 0$ and a tangential step is computed, condition (2.22) ensures that the computed tangential direction is nonzero. However, since $t_k = 0$ we must have redefined t_k to be zero because the computed tangential direction satisfied (2.30) but not (2.31). This is a contradiction because (2.31) would have been satisfied trivially since $n_k = 0$. Thus (2.27) must fail. Since we have shown that both (2.28) and (2.27) must fail at y-iterations for which x_k is not an infeasible stationary point, we conclude that $\pi_k \leq \omega_t(\|c_k\|) \leq \omega_t(\omega_n(\pi_{k-1}))$ where we used the monotonicity of ω_t. The desired conclusion follows from the second part of (2.29). □

We conclude this section by stating the basic property of the step length.

Lemma 2.4 *We have that, for all k,*

$$\|s_k\| \leq \Delta_k \leq \Delta_k^c \qquad \text{if } t_k \neq 0 \tag{2.47}$$

while

$$\|s_k\| = \|n_k\| \leq \Delta_k^c \qquad \text{if } t_k = 0. \tag{2.48}$$

Proof. If $t_k \neq 0$ is computed, then (2.13) holds and (2.11) and (2.12) together give the bound (2.47). In the other case, $s_k = n_k$ and (2.3) ensures (2.48). □

3 Global convergence to first-order critical points

3.1 Assumptions and preliminaries

Before starting our convergence analysis, we recall our assumption that both f and c are twice continuously differentiable. Moreover, we also assume that there exists a constant $\kappa_{\rm H}$ such that, for all ξ in $\bigcup_{k\geq 0}[x_k, x_k^+] \cup \mathcal{L}$, all k and all $i \in \{1, \ldots, m\}$,

$$1 + \max\left[\, \|g_k\|, \|\nabla_{xx} f(\xi)\|, \|\nabla_{xx} c_i(\xi)\|, \|J(\xi)\|, \|H_k\|, \|C_{ik}\|\,\right] \leq \kappa_{\rm H}. \tag{3.1}$$

When H_k and C_{ik} are chosen as $\nabla_{xx} f(x_k)$ and $\nabla_{xx} c_i(x_k)$, respectively, this last assumption is for instance satisfied if the first and second derivatives of f and c are uniformly bounded, or, because of continuity, if the sequences $\{x_k\}$ and $\{x_k^+\}$ remain in a bounded domain of $\mathbb{R}^n$.

We finally complete our set of assumptions by supposing that

$$f(x) \geq f_{\rm low} \quad \text{for all} \quad x \in \mathcal{L}. \tag{3.2}$$

This assumption is often realistic and is, for instance, satisfied if the smallest singular value of the constraint Jacobian $J(x)$ is uniformly bounded away from zero. Observe that (3.2) obviously holds by continuity if we assume that all iterates remain in a bounded domain.

We first state some useful consequences of (3.1).

Lemma 3.1 *For all k,*

$$1 + \|W_k\| \leq \kappa_{\rm H}^2, \tag{3.3}$$

and

$$\|g_k^N\| \;\leq [1 + \kappa_{\rm n}\sqrt{2\theta_0^{\max}}(1 + m\kappa_{\rm y})]\kappa_{\rm H} \stackrel{\rm def}{=} \kappa_{\rm g} \tag{3.4}$$

Proof. The first inequality immediately follows from

$$1 + \|W_k\| = 1 + \|J_k\|^2 \leq (1 + \|J_k\|)^2 \leq \kappa_{\rm H}^2,$$

where the last inequality is deduced from (3.1). The bound (3.4) is obtained from (2.7), the inequality

$$\|g_k^{\rm N}\| \leq \|g_k\| + \|G_k\|\,\|n_k\| \leq \|g_k\| + \kappa_{\rm n}\big[\|H_k\|\,\|c_k\| + m\|\hat{y}_k\|\,\|c_k\| \max_{i=1,\ldots,m} \|C_{i,k}\|\big],$$

Lemma 2.2, (2.9) and (3.1). □

We also establish a useful sufficient condition for first-order criticality.

Lemma 3.2 *Suppose that for some infinite subsequence indexed by $\mathcal{K}$ we have*

$$\lim_{k\to\infty, k\in\mathcal{K}} \|c_k\| = 0. \tag{3.5}$$

Then

$$\lim_{k\to\infty, k\in\mathcal{K}} g_k^N = \lim_{k\to\infty, k\in\mathcal{K}} g_k. \tag{3.6}$$

If, in addition,

$$\lim_{k\to\infty, k\in\mathcal{K}} \pi_k = 0, \tag{3.7}$$

then

$$\lim_{k\to\infty, k\in\mathcal{K}} g_k + J_k^T y_k = 0 \quad \textit{and} \quad \lim_{k\to\infty, k\in\mathcal{K}} \|P_k g_k\| = 0, \tag{3.8}$$

where P_k is the orthogonal projection onto the nullspace of J_k, and all limit points of the sequence $\{x_k\}_{k\in\mathcal{K}}$ (if any) are first-order critical.

Proof. Combining the uniform bound (3.4) with (2.15), we obtain that the sequence $\{\|r_k\|\}_{\mathcal{K}}$ is uniformly bounded and therefore can be considered as the union of convergent subsequences. Moreover, because of (2.5), the limit (3.5) first implies that

$$\lim_{k\to\infty, k\in\mathcal{K}} n_k = 0, \tag{3.9}$$

which then implies with (2.7) and (3.1) that (3.6) holds. This limit, together with (2.14) and (2.17), ensures that

$$\lim_{k\to\infty, k\in\mathcal{P}} r_k = \lim_{k\to\infty, k\in\mathcal{P}} [g_k + J_k^T y_k] = \lim_{k\to\infty, k\in\mathcal{P}} [g_k - J_k^T [J_k^T]^I g_k] = \lim_{k\to\infty, k\in\mathcal{P}} P_k g_k, \tag{3.10}$$

where we have restricted our attention on a particular subsequence indexed by $\mathcal{P} \subseteq \mathcal{K}$ such that the limit in the left-hand side is well-defined. Assume now that this limit is a nonzero vector. Then, using now (2.21), (3.9), (3.6) and the Hermitian and idempotent nature of P_k, we have that

$$\begin{aligned} \lim_{k\to\infty, k\in\mathcal{P}} \pi_k &= \lim_{k\to\infty, k\in\mathcal{P}} \frac{\langle g_k, r_k\rangle}{\|r_k\|} = \lim_{k\to\infty, k\in\mathcal{P}} \frac{\langle g_k, P_k g_k\rangle}{\|P_k g_k\|} \\ &= \lim_{k\to\infty, k\in\mathcal{P}} \frac{\langle P_k g_k, P_k g_k\rangle}{\|P_k g_k\|} = \lim_{k\to\infty, k\in\mathcal{P}} \|P_k g_k\|. \end{aligned} \tag{3.11}$$

But (3.7) implies that this latter limit is zero, and (3.10) also gives that r_k must converge to zero along $\mathcal{P}$, which is impossible. Hence $\lim_{k\to\infty, k\in\mathcal{P}} r_k = 0$ and the desired conclusion then follows from (3.10). □

This lemma indicates that all we need to show for first-order global convergence are the two limits (3.5) and (3.7) for an index set $\mathcal{K}$ as large as possible. Unfortunately, and as is unavoidable with local methods for constrained optimization, our algorithm may fail to produce (3.5)–(3.7) and, instead, end up being trapped by a local infeasible stationary point of the infeasibility measure $\theta(x)$. If $x_\diamond$ is such a point, then

$$J(x_\diamond)^T c(x_\diamond) = 0 \quad \text{with} \quad c(x_\diamond) \neq 0.$$

If started from $x_\diamond$, Algorithm 2.1 will fail to progress towards feasibility, as no suitable normal step can be found in Step 2. A less unlikely scenario, where there exists a subsequence indexed by $\mathcal{Z}$ such that

$$\lim_{k\to\infty, k\in\mathcal{Z}} \|J_k^T c_k\| = 0 \quad \text{with} \quad \liminf_{k\to\infty, k\in\mathcal{Z}} \|c_k\| > 0, \tag{3.12}$$

indicates the approach of such an infeasible stationary point. In both cases, restarting the whole algorithm from a different starting point might be the best strategy. Barring this undesirable situation, we would however like to show that our algorithm converges to first-order critical points for (1.1), whenever uniform asymptotic convexity of $\theta(x)$ in the range space of J_k is obtained when feasibility is approached. More specifically, we assume from now on that, for some small constant $\kappa_{\rm c} \in (0, 1)$,

$$\text{there exists } \kappa_{\rm J} \in (0,1) \text{ such that } \sigma_{\min}(J_k) \geq \kappa_{\rm J} \text{ whenever } \|c(x_k)\| \leq \kappa_{\rm c}, \tag{3.13}$$

where $\sigma_{\min}(A)$ is the smallest positive singular value of the matrix A. It is important to note that this assumption holds by continuity if $J(x)$ is Lipschitz continuous and $\sigma_{\min}(J(x))$ uniformly bounded away from zero on the feasible set, in which case the Jacobian of the constraints has constant rank over this set.

We also obtain the following useful bound.

Lemma 3.3 *There exists a constant $\kappa_{\rm G} > \kappa_{\rm H}$ such that, $1 + \|G_k\| \leq \kappa_{\rm G}$ for every k.*

Proof. The desired conclusion follows from (2.8), (2.9) and (3.1), with

$$\kappa_{\rm G} \stackrel{\rm def}{=} \kappa_{\rm H} + m\kappa_{\rm H}\sqrt{2\theta_0^{\max}} > \kappa_{\rm H}.$$

□

The following two simple properties result from the mechanism of the algorithm.

Lemma 3.4 *For all $k \in \mathcal{C} \cap \mathcal{S}$, one has that*

$$\|t_k\| \leq \kappa_{\mathcal{CS}} \|n_k\|. \tag{3.14}$$

Proof. Consider $k \in \mathcal{C} \cap \mathcal{S}$. Hence $\theta(x_k^+) < \theta(x_k)$ and, in view of Lemma 2.2, (2.33) holds. If (3.14) fails, then $t_k \neq 0$, the mechanism of the algorithm implies that (2.31) must also hold, and thus that $k \in \mathcal{F}$ which is impossible. Hence (3.14) must hold. □

Lemma 3.5 *We have the following results related to the set $\mathcal{A}$.*

(i) *If $k \notin \mathcal{Y}$ and $\|t_k\| \leq \kappa_{\rm gen}\|n_k\|$ for some generic constant $\kappa_{\rm gen} > 0$, then $k \in \mathcal{A}$.*

(ii) *$\mathcal{C} \cap \mathcal{S} \subseteq \mathcal{A}$.*

Proof. We first prove part (i). Assume that $n_k = 0$. The given assumptions then imply that $t_k = 0$ and, therefore, that $k \in \mathcal{Y}$. This is a contradiction so we may conclude that $n_k \neq 0$ and thus $k \in \mathcal{A}$.

We now prove part (ii). Let $k \in \mathcal{C} \cap \mathcal{S}$ from which it follows that $k \notin \mathcal{Y}$. Moreover, Lemma 3.4 implies that $\|t_k\| \leq \kappa_{\mathcal{CS}}\|n_k\|$. We may now apply part (i) with $\kappa_{\rm gen} = \kappa_{\mathcal{CS}}$ to conclude that $k \in \mathcal{A}$. □

As for most of the existing theory for convergence of trust-region methods, we also make use of the following direct consequences of Taylor's theorem.

Lemma 3.6 *For all $k \in \mathcal{F}$, we have that*

$$|f(x_k^+) - m_k(x_k^+)| \leq \kappa_{\rm G}\Delta_k^2, \tag{3.15}$$

while, for all k,

$$|\, \|c(x_k^+)\|^2 - \|c_k + J_k s_k\|^2| \leq 2\kappa_{\rm C}[\Delta_k^c]^2, \tag{3.16}$$

with $\kappa_{\rm C} = \kappa_{\rm H}^2 + m\kappa_{\rm H}\sqrt{2\theta_0^{\max}} > \kappa_{\rm H}$, and

$$|\theta(x_k^+) - \tfrac{1}{2}\|c_k + J_k s_k\|^2| \leq \kappa_{\theta 1}\|s_k\|^3 + \kappa_{\theta 2}\|c_k\|\|s_k\|^2, \tag{3.17}$$

with $\kappa_{\theta 1} = (m + \frac{1}{2}\sqrt{m})\,\kappa_{\rm H}^2$ and $\kappa_{\theta 2} = \frac{1}{2}\kappa_{\rm H}\sqrt{m}$.

Proof. The inequality (3.15) follows from Lemma 3.3, the fact that $f(x)$ is twice continuously differentiable and Lemma 2.4 (see Theorem 6.4.1 in Conn et al., 2000). Similarly, (3.16) follows from the fact that $\theta(x)$ is twice continuously differentiable with its Hessian given by

$$\nabla_{xx}\theta(x) = J(x)^T J(x) + \sum_{i=1}^{m} c_i(x)\nabla_{xx}c_i(x), \tag{3.18}$$

(3.1), Lemma 2.2 and Lemma 2.4.

We now prove (3.17). Using the mean-value theorem, we obtain that

$$\theta(x_k^+) = \theta_k + \langle J_k^T c_k, s_k\rangle + \tfrac{1}{2}\langle s_k, \nabla_{xx}\theta(\xi_k)s_k\rangle$$

for some $\xi_k \in [x_k, x_k^+]$, which implies, in view of (3.18), that

$$\left|\theta(x_k^+) - \theta_k - \langle c_k, J_k s_k\rangle - \tfrac{1}{2}\|J(\xi_k)s_k\|^2\right| = \tfrac{1}{2}\left|\sum_{i=1}^{m} c_i(\xi_k)\langle s_k, \nabla_{xx}c_i(\xi_k)s_k\rangle\right|. \tag{3.19}$$

A further application of the mean-value theorem then gives that

$$c_i(\xi_k) = c_i(x_k) + \langle e_i, J(\mu_{k,i})(\xi_k - x_k)\rangle = c_i(x_k) + \langle J(\mu_{k,i})^T e_i, \xi_k - x_k\rangle$$

for some $\mu_{k,i} \in [x_k, \xi_k]$. Summing over all constraints and using the triangle inequality, (3.1) (twice) and the bound $\|\xi_k - x_k\| \le \|s_k\|$, we thus obtain that

$$\begin{aligned}\left|\sum_{i=1}^m c_i(\xi_k)\langle s_k, \nabla_{xx} c_i(\xi_k) s_k\rangle\right| &\le \left[\,\|c_k\|_1 + \kappa_{\rm H}\sqrt{m}\|s_k\|\,\right]\kappa_{\rm H}\|s_k\|^2\\ &\le \kappa_{\rm H}\sqrt{m}\,\|c_k\|\,\|s_k\|^2 + \kappa_{\rm H}^2\sqrt{m}\|s_k\|^3\end{aligned}$$

Substituting this inequality into (3.19), we deduce that

$$\begin{aligned}|\theta(x_k^+) - \tfrac{1}{2}\|c_k + J_k s_k\|^2| &\le \tfrac{1}{2}\left|\,\|J(\xi_k)s_k\|^2 - \|J_k s_k\|^2\right|\\ &\quad + \tfrac{1}{2}\kappa_{\rm H}\sqrt{m}\,\|c_k\|\,\|s_k\|^2 + \tfrac{1}{2}\kappa_{\rm H}^2\sqrt{m}\|s_k\|^3\end{aligned} \tag{3.20}$$

Define now $\phi_k(x) \stackrel{\rm def}{=} \frac{1}{2}\|J(x)s_k\|^2$. Then a simple calculation shows that

$$\nabla_x \phi_k(x) = \sum_{i=1}^m [J(x)s_k]_i \nabla_{xx} c_i(x) s_k.$$

Using this relation, the mean-value theorem again and (3.1), we obtain that

$$\begin{aligned}|\phi_k(\xi_k) - \phi_k(x_k)| &= |\langle \xi_k - x_k, \nabla_x \phi_k(\zeta_k)\rangle|\\ &= |\langle \xi_k - x_k, \textstyle\sum_{i=1}^m [J(\zeta_k)s_k]_i \nabla_{xx} c_i(\zeta_k) s_k\rangle|\\ &\le \sum_{i=1}^m \|\xi_k - x_k\|\,\|\nabla_{xx} c_i(\zeta_k)\|\,\|J(\zeta_k)\|\,\|s_k\|^2\\ &\le m\,\kappa_{\rm H}^2\|s_k\|^3\end{aligned}$$

for some $\zeta_k \in [x_k, \xi_k] \subseteq [x_k, x_k + s_k]$. We therefore obtain that

$$\tfrac{1}{2}\left|\,\|J(\xi_k)s_k\|^2 - \|J_k s_k\|^2\right| = |\phi_k(\xi_k) - \phi_k(x_k)| \le m\,\kappa_{\rm H}^2\|s_k\|^3. \tag{3.21}$$

We then obtain (3.17) using (3.20) and (3.21), (2.11) successively. □

The third conclusion of this lemma also allows us to deduce that all c-iterations are in $\mathcal{C}_t$ for Δ_k^c sufficiently small.

Lemma 3.7 *Suppose that $k \in \mathcal{C}$ and that*

$$\Delta_k^c \le \frac{(1-\kappa_{\rm tt})}{\kappa_\Delta(\kappa_{\theta 1}\kappa_\Delta + \kappa_{\theta 2}\sqrt{2})} \stackrel{\rm def}{=} \kappa_{\mathcal{C}}. \tag{3.22}$$

Then $k \in \mathcal{C}_t$.

Proof. Assume that $k \in C_w$. Then, using (3.17), (2.1), Lemma 2.2, (2.25), (2.24), Lemma 2.4 and (3.22) successively, we obtain that

$$\begin{aligned}\theta(x_k^+) &\le \kappa_{\rm tt}\theta_k^{\max} + \kappa_{\theta 1}\|s_k\|^3 + \kappa_{\theta 2}\sqrt{2\theta_k^{\max}}\,\|s_k\|^2\\ &\le \kappa_{\rm tt}\theta_k^{\max} + \kappa_{\theta 1}\kappa_\Delta^2\theta_k^{\max}\Delta_k^c + \kappa_\Delta\kappa_{\theta 2}\sqrt{2}\,\theta_k^{\max}\Delta_k^c\\ &\le \theta_k^{\max}.\end{aligned} \tag{3.23}$$

This implies that (2.33) holds. On the other hand, the fact that $k \in \mathcal{C}_w$ ensures that (2.23) holds, and thus, since $\bar{\kappa}_\delta$ satisfies $\bar{\kappa}_\delta = 1/(1-\kappa_\delta)$ by definition, that (2.31) also holds. Combining these observations and noting that a tangential step was computed at iteration k since $k \in \mathcal{C}_w$ by assumption, we obtain that $k \in \mathcal{F}$, which is a contradiction because $k \in \mathcal{C}$. Hence our assumption that $k \in \mathcal{C}_w$ is impossible and the desired conclusion follows. □

Lemmas 3.6 and 3.7 have the following useful consequence.

Lemma 3.8 *Suppose that $k \in \mathcal{C}$ and*

$$[\Delta_k^c]^2 \le \min\left[1, \kappa_{\mathcal{C}}^2, \frac{\frac{1}{2}(1-\kappa_{\theta\theta})\left(\frac{\|J_k^T c_k\|}{\kappa_{\rm H}}\right)^2}{\kappa_{\theta 1} + \kappa_{\theta 2}\sqrt{2\theta_0^{\max}}}\right] \stackrel{\rm def}{=} \left(\min[\kappa_{{\rm n}\Delta 1}, \kappa_{{\rm n}\Delta 2}\|J_k^T c_k\|]\right)^2 \tag{3.24}$$

Then $k \in \mathcal{A}$ and $n_k \neq 0$.

Proof. Assume that $k \in \mathcal{C}$, $n_k = 0$ and (3.24) holds. Observe first that the mechanism of the algorithm ensures that $\Delta_k^c > 0$ for all k, and thus (3.24) implies that $\|J_k c_k\| > 0$. Now (3.24) and Lemma 3.7 imply that $k \in \mathcal{C}_t$. Thus, in view of (2.4), our assumption that $n_k = 0$ can only hold if n_k is not computed to satisfy (2.4) at iteration k, and (2.28) must therefore fail. This in turn implies that

$$\theta_k \le \kappa_{\theta\theta}\theta_k^{\max}.$$

This bound, (3.17), (2.26), (2.1) and Lemmas 2.2 and 2.4 then give that

$$\theta(x_k^+) \le \kappa_{\theta\theta}\theta_k^{\max} + \kappa_{\theta 1}[\Delta_k^c]^3 + \kappa_{\theta 2}\sqrt{2\theta_0^{\max}}[\Delta_k^c]^2,$$

which, with (3.24), yields that

$$\begin{aligned}\theta(x_k^+) &\le \kappa_{\theta\theta}\theta_k^{\max} + (\kappa_{\theta 1} + \kappa_{\theta 2}\sqrt{2\theta_0^{\max}})[\Delta_k^c]^2 \\ &\le \kappa_{\theta\theta}\theta_k^{\max} + (1-\kappa_{\theta\theta})\frac{\|J_k^T c_k\|^2}{2\kappa_{\rm H}^2}.\end{aligned} \tag{3.25}$$

But (2.46) and (3.1) imply that

$$\theta_k^{\max} \ge \theta_k \ge \frac{\|J_k^T c_k\|^2}{2\kappa_{\rm H}^2}.$$

Combining this last inequality with (3.25), we obtain that $\theta(x_k^+) < \theta_k^{\max}$ and (2.33) holds. Now observe that we must have that $t_k \neq 0$ (otherwise iteration k would be a y-iteration). Moreover, (2.31) trivially holds since $\delta_k^f = \delta_k^{f,t}$. But this in turn implies that $k \in \mathcal{F}$, which is a contradiction, and thus $n_k \neq 0$ and $k \in \mathcal{A}$. □

3.2 A "limit inferior" result

We now investigate the relation between the trust-region radii and their associated criticality measures.

Lemma 3.9 *If $k \in \mathcal{F}$ and*

$$\Delta_k \le \frac{\kappa_\delta \kappa_{\rm tC}\pi_k(1-\eta_2)}{\kappa_{\rm G}}, \tag{3.26}$$

then $\rho_k^f \ge \eta_2$, iteration k is very successful and $\Delta_{k+1}^f \ge \Delta_k^f$. Similarly, if $k \in \mathcal{C}$ and

$$\Delta_k^c \le \min\left[\kappa_{\mathcal{C}}, \frac{\kappa_{\rm nC2}\|J_k^T c_k\|(1-\eta_2)}{\kappa_{\rm H}^2}, \kappa_{{\rm n}\Delta 1}, \kappa_{{\rm n}\Delta 2}\|J_k^T c_k\|\right] \stackrel{\rm def}{=} \min\left[\kappa_{\Delta{\rm c}1}, \kappa_{\Delta{\rm c}2}\|J_k^T c_k\|\right], \tag{3.27}$$

then $k \in \mathcal{A}$, $n_k \neq 0$, $\rho_k^c \ge \eta_2$, iteration k is very successful and $\Delta_{k+1}^c \ge \Delta_k^c$.

Proof. The proof of both statements is identical to that of Theorem 6.4.2 of Conn et al. (2000) for the objective functions $f(x)$ and $\theta(x)$, respectively. In the first case, one uses (2.22), (2.31) and (3.15). For proving the second statement, we first observe that $J_k^T c_k \neq 0$ since $\Delta_k^c > 0$. Furthermore, (3.27) and Lemma 3.8 together imply that $k \in \mathcal{A}$ and $n_k \neq 0$. One then notices that (3.27) also implies, in view of Lemma 3.7, that $k \in \mathcal{C}_t$ and thus, because of Lemma 2.1 that the first inequality of (2.39) and (2.44) hold. This last inequality in then used together with (3.1), (3.16) and the bound (3.3) to deduce the second conclusion. □

The mechanism for updating the trust-region radii then implies the next crucial lemmas, where we show that the radius of either trust region cannot become arbitrarily small compared to the considered criticality measure for dual and primal feasibility.

Lemma 3.10 *Suppose that, for some* $\epsilon_f > 0$,

$$\pi_k \geq \epsilon_f \text{ for all } k \in \mathcal{F}. \tag{3.28}$$

Then, for all k,

$$\Delta_k^f \geq \gamma_1 \min\left[\frac{\kappa_\delta \kappa_{\rm tC} \epsilon_f (1-\eta_2)}{\kappa_{\rm G}}, \Delta_0^f\right] \stackrel{\rm def}{=} \epsilon_{\mathcal{F}}. \tag{3.29}$$

Proof. The statement immediately results from the mechanism of the algorithm, Lemma 3.9 and the inequality $\Delta_k \leq \Delta_k^f$, given that Δ_k^f is only decreased at f-iterations. □

Lemma 3.11 *We have that, for all* k,

$$\Delta_k^c \geq \min\left[\kappa_{\Delta c3}, \kappa_{\Delta c4}\|J_k^T c_k\|\right] \tag{3.30}$$

for some $\kappa_{\Delta c3}$ *and* $\kappa_{\Delta c4}$ *both in* $(0,1)$*. In particular, if we assume that, for some* $\epsilon_\theta > 0$,

$$\|J_k^T c_k\| \geq \epsilon_\theta \text{ for all } k \in \mathcal{C}, \tag{3.31}$$

then, for all k,

$$\Delta_k^c \geq \gamma_1 \min\left[\kappa_{\Delta c3}, \kappa_{\Delta c4}\epsilon_\theta\right] \stackrel{\rm def}{=} \epsilon_{\mathcal{C}}. \tag{3.32}$$

Proof. Assume that, at iteration k,

$$\Delta_k^c \geq \gamma_1 \min[\Delta_0^c, \kappa_{\Delta c1}, \kappa_{\Delta cc}, \kappa_{\Delta c2}\|J_k^T c_k\|], \tag{3.33}$$

and note that this assumption is obviously verified for $k = 0$. We now distinguish different cases depending on the nature of iteration k. Assume first that $k \in \mathcal{Y} \cup (\mathcal{F} \setminus \mathcal{S})$. Since Δ_k^c is unmodified and $x_{k+1} = x_k$ at such an iteration, we conclude that (3.33) again holds at iteration $k+1$. Assume next that $k \in \mathcal{F} \cap \mathcal{S}$. Then (2.37) ensures that (3.33) also holds at iteration $k+1$ since $\gamma_1 < 1$. Similarly, (2.40) ensure that (3.33) holds at iteration $k+1$ if $k \in \mathcal{C} \cap \mathcal{S}$. Assume finally that $k \in \mathcal{C} \setminus \mathcal{S}$. In this case, the second part of Lemma 3.9 implies that (in addition to (3.33))

$$\Delta_k^c \geq \min[\kappa_{\Delta c1}, \kappa_{\Delta c2}\|J_k^T c_k\|],$$

and (2.40) and the identity $x_{k+1} = x_k$ then imply that

$$\Delta_{k+1}^c \geq \gamma_1 \min[\Delta_0^c, \kappa_{\Delta c1}, \kappa_{\Delta cc}, \kappa_{\Delta c2}\|J_{k+1}^T c_{k+1}\|].$$

Thus (3.33) again holds at iteration $k+1$. We therefore obtain that (3.30) holds for all $k \geq 0$ with $\kappa_{\Delta c3} = \gamma_1 \min[\Delta_0^c, \kappa_{\Delta c1}, \kappa_{\Delta cc}]$ and $\kappa_{\Delta c4} = \gamma_1 \kappa_{\Delta c2}$. The bound (3.32) then directly follows from (3.30), (3.31) and the observation that Δ_k^c can only be decreased at c-iterations. □

We now start our analysis proper by considering the case where the number of successful iterations is finite.

Lemma 3.12 *Suppose that* $|\mathcal{S}| < +\infty$*. Then there exists an* x_* *and a* y_* *such that* $x_k = x_*$ *and* $y_k = y_*$ *for all sufficiently large* k*, and either*

$$J(x_*)^T c(x_*) = 0 \quad \textit{and} \quad c(x_*) \neq 0,$$

or

$$P_* g(x_*) = 0 \quad \textit{and} \quad c(x_*) = 0,$$

where P_* *is the orthogonal projection onto the nullspace of* $J(x_*)$*.*

Proof. The existence of a suitable x_* immediately results from the mechanism of the algorithm and the finiteness of $\mathcal{S}$, which implies that $x_* = x_{k_s+j}$ for all $j \geq 1$, where k_s is the index of the last successful iteration.

Assume first that there are infinitely many c-iterations. This yields that Δ_k^c is decreased in (2.40) at every such iteration for $k \geq k_s$ and therefore that $\{\Delta_k^c\}$ converges to zero, because it is never increased at y-iterations or unsuccessful f-iterations. Lemma 3.7 then implies that all c-iterations are in $\mathcal{C}_t$ for k large enough. Since, for such a k, $\|J_k^T c_k\| = \|J(x_*)^T c(x_*)\|$ for all $k > k_s$, this in turn implies, in view of the Lemma 3.11, that $\|J(x_*)^T c(x_*)\| = 0$. If x_* is not feasible, then we obtain the first of the two possibilities listed in the lemma's statement. On the other hand, if $c(x_*) = 0$ then we have from (2.5) that $n_k = 0$, and thus that $\delta_k^f = \delta_k^{f,t} \geq 0$ for all k sufficiently large. Hence (2.31) holds for k large. Moreover, we also obtain from (2.26) (which must hold for k large because $\mathcal{C}$ is asymptotically equal to $\mathcal{C}_t$) that $\|c_k + J_k s_k\| = 0$ and also, since $\theta_k^{\max}$ is only reduced at successful c-iterations, that $\theta_k^{\max} = \theta_*^{\max} > 0$ for all k sufficiently large. Combining these observations, we then obtain from Lemma 3.6 that

$$\theta(x_k^+) = \theta(x_k^+) - \tfrac{1}{2}\|c_k + J_k s_k\|^2 \leq \kappa_{\rm C}[\Delta_k^c]^2 \leq \theta_*^{\max} = \theta_k^{\max},$$

and thus (2.33) holds for all sufficiently large k. We have that t_k must be zero for all $k \in \mathcal{C}$ sufficiently large (otherwise iteration k would be a f-iteration). Since we already know that $n_k = 0$ for all k large enough, we thus obtain that $s_k = 0$ for these k and must eventually be y-iterations, which is a contradiction. Hence our assumption that there are infinitely many c-iterations is impossible.

Assume now that $\mathcal{C}$ is finite but $\mathcal{F}$ infinite. Since there must be an infinite number of unsuccessful f-iterations after k_s, and since the radii are not updated at y-iterations, we obtain that $\{\Delta_k^f\}$, and hence $\{\Delta_k\}$, converge to zero. Using now Lemma 3.10, we conclude that $\pi_k = 0$ along some infinite subsequence, and because (2.27) holds at f-iterations, that $\|c_k\| = 0$ along that subsequence. Thus $c(x_*) = 0$. As above, the second of the lemma's statements then holds because of this equality, the fact that $\pi_k = 0$ for for an infinite subsequence of k with $g_k = g_*$ and $P_k = P_*$ and Lemma 3.2.

Assume finally that $\mathcal{C} \cup \mathcal{F}$ is finite. Thus all iterations must be y-iterations for k large enough. In view of Lemma 2.3, we must have $\pi_* = 0$. If $c(x_*) = 0$, we are done because of Lemma 3.2. Otherwise, we know from Lemma 2.3, (2.29), and (2.28) that we must have computed a normal step n_k, but since by assumption $n_k = t_k = 0$, we must have that $J_k^T c_k = 0$. Hence $J(x_*)^T c(x_*) = 0$ and the first part of the lemma holds. □

We now turn to the more complicated case where there are infinitely many successful iterations, and start by proving a result directly inspired by Lemma 6.5.1 of Conn et al. (2000) and using $s_k^R \stackrel{\rm def}{=} (I - P_k)s_k$, the projection of s_k onto the range space of J_k^T.

Lemma 3.13 *Suppose that* $k \in \mathcal{A} \cap \mathcal{C}_t$*, and that*

$$\|c_k\| \leq \kappa_{\rm c}. \tag{3.34}$$

Then

$$\|s_k^R\| \le \frac{2}{\kappa_{\rm J}^2}\|J_k^T c_k\| \tag{3.35}$$

and

$$\delta_k^c \ge \kappa_{\rm R}\|s_k^R\|^2, \tag{3.36}$$

where

$$\kappa_{\rm R} \stackrel{\rm def}{=} \tfrac{1}{2}\kappa_{\rm J}^2\kappa_{\rm nC2}\min\left[\frac{\kappa_{\rm J}^2}{2\kappa_{\rm H}^2}, 1\right]. \tag{3.37}$$

Proof. The proof of (3.35) is identical to that of Lemma 6.5.1 in Conn et al. (2000) (applied on the minimization of $\theta(x)$ in the range space of J_k^T using the model (2.2)), taking into account that the smallest eigenvalue of W_k is bounded below by $\kappa_{\rm J}^2$ because of (3.34) and (3.13). Substituting now (3.35) in (2.44) (which must hold since $k \in \mathcal{A} \cap \mathcal{C}_t$) and using (3.3) then yields that

$$\delta_k^c \ge \tfrac{1}{2}\kappa_{\rm J}^2\kappa_{\rm nC2}\|s_k^R\|\min\left[\frac{\kappa_{\rm J}^2\|s_k^R\|}{2\kappa_{\rm H}^2}, \Delta_k^c\right],$$

which, using the bound $\|s_k^R\| \le \|s_k\| \le \Delta_k^c$, gives (3.36) with (3.37). □

Our analysis now focuses on unsuccessful c-iterations (the only one at which Δ_k^c can be decreased) and first consider what happens if the constraints' violation is small enough.

Lemma 3.14 *Suppose that $k \in \mathcal{A} \cap \mathcal{C}_t \setminus \mathcal{S}$ such that*

$$\|c_k\| \le \min\left[\kappa_{\rm c}, \frac{\kappa_{\Delta {\rm c}1}}{\kappa_{\Delta{\rm c}2}\kappa_{\rm J}}\right]. \tag{3.38}$$

Then

$$\|c_k + J_k s_k\|^2 \le \kappa_{\rm cld}\|c_k\|^2 \tag{3.39}$$

and

$$\|s_k^R\| \ge \kappa_{\rm sRn}\|n_k\| \tag{3.40}$$

with

$$\kappa_{\rm cld} \stackrel{\rm def}{=} 1 - 2(1-\kappa_{\rm tg})\kappa_{\rm nC}\kappa_{\rm J}^2\min\left[\frac{1}{\kappa_{\rm H}^2}, \kappa_{\Delta{\rm c}2}\right] \in (0,1) \;\; \textit{and} \;\; \kappa_{\rm sRn} \stackrel{\rm def}{=} (1-\sqrt{\kappa_{\rm cld}})/\kappa_{\rm n}\kappa_{\rm H} \in (0,1).$$

Proof. Since $k \in \mathcal{A} \cap \mathcal{C}_t$,, we may use Lemma 2.1, (3.38) and (3.13) to obtain

$$\begin{aligned} \|c_k + J_k(n_k+t_k)\|^2 &\le \|c_k\|^2 - 2\kappa_{\rm nC2}\|J_k^T c_k\|\min\left[\frac{\|J_k^T c_k\|}{\kappa_{\rm H}^2}, \Delta_k^c\right] \\ &\le \|c_k\|^2 - 2\kappa_{\rm nC2}\kappa_{\rm J}\|c_k\|\min\left[\kappa_{\rm J}\frac{\|c_k\|}{\kappa_{\rm H}^2}, \Delta_k^c\right]. \end{aligned} \tag{3.41}$$

But we know from Lemma 3.9 that, at unsuccessful c-iterations,

$$\Delta_k^c > \min\left[\kappa_{\Delta{\rm c}1}, \kappa_{\Delta{\rm c}2}\|J_k^T c_k\|\right]$$

and hence

$$\Delta_k^c \ge \min\left[\kappa_{\Delta{\rm c}1}, \kappa_{\Delta{\rm c}2}\kappa_{\rm J}\|c_k\|\right] \ge \kappa_{\Delta{\rm c}2}\kappa_{\rm J}\|c_k\|,$$

where we again used (3.38) and (3.13). Substituting this last bound in (3.41) then yields that

$$\|c_k + J_k s_k\|^2 \le \|c_k\|^2 - 2\kappa_{\rm nC2}\kappa_{\rm J}\|c_k\|\min\left[\frac{\kappa_{\rm J}}{\kappa_{\rm H}^2}\|c_k\|, \kappa_{\Delta{\rm c}2}\kappa_{\rm J}\|c_k\|\right] \le \kappa_{\rm cld}\|c_k\|^2.$$

Note that $\kappa_{\rm cld} \in (0,1)$. We have therefore proved the first statement of the lemma. Using the definition of s_k^R and the reverse triangle inequality, we may also deduce that, for $k \in \mathcal{K}$ sufficiently large,

$$\|c_k\| - \|J_k s_k^R\| \leq \|c_k + J_k s_k^R\| \leq \sqrt{\kappa_{\rm cld}}\|c_k\|,$$

and therefore, using (3.1), that, for such k,

$$\kappa_{\rm H}\|s_k^R\| \geq \|J_k s_k^R\| \geq (1-\sqrt{\kappa_{\rm cld}})\|c_k\| \geq \frac{1-\sqrt{\kappa_{\rm cld}}}{\kappa_{\rm n}}\|n_k\|,$$

where we used (2.5) to deduce the last inequality. This in turn yields (3.40). □

We now distinguish the case where $\|n_k\|$ is small with respect to $\|t_k\|$ at unsuccessful c-iterations from the case where it is large. We start by considering the former.

Lemma 3.15 *Suppose that $k \notin \mathcal{Y}$ and that*

$$\pi_k \geq \epsilon_f > 0, \tag{3.42}$$

$$\Delta_k^c \leq \min\left[1, \kappa_{\mathcal{C}}, \frac{\epsilon_f}{\kappa_{\rm G}}\right] \stackrel{\rm def}{=} \kappa_{\delta\mathcal{C}} \tag{3.43}$$

and

$$\|t_k\| \geq \varsigma(\epsilon_f)\|n_k\|, \tag{3.44}$$

where

$$\varsigma(\epsilon) \stackrel{\rm def}{=} \max\left[\frac{2\kappa_{\rm G}}{(1-\kappa_\delta)(\kappa_{\mathcal{CS}}-1)\kappa_{\rm tC}\epsilon}, 1\right]\kappa_{\mathcal{CS}}.$$

Then $t_k \neq 0$ and (2.31) holds.

Proof. First note that the conclusions of the lemma hold by definition if $k \in \mathcal{F}$. Suppose therefore that $k \in \mathcal{C}$, and consider first the case where $n_k = 0$. We must then have that $t_k \neq 0$, otherwise iteration k would be a y-iteration. Moreover $\delta_k^f = \delta_k^{f,t}$ and thus (2.31) holds.

Suppose now that $n_k \neq 0$. Because (3.44) holds, we have that $t_k \neq 0$. Now we have, on one hand, that

$$\Delta_k \geq \|s_k\| \geq \|t_k\| - \|n_k\| = \|t_k\|\left(1 - \frac{\|n_k\|}{\|t_k\|}\right) \geq \left(1 - \frac{1}{\kappa_{\mathcal{CS}}}\right)\|t_k\| \tag{3.45}$$

where we have successively applied (2.47), the reverse triangle inequality and (3.44). On the other hand, we know that

$$-\delta_k^{f,n} = \langle g_k, n_k\rangle + \tfrac{1}{2}\langle n_k, G_k n_k\rangle$$

and we may therefore deduce from the Cauchy-Schwarz inequality, (3.1) and Lemma 3.3 that

$$|\delta_k^{f,n}| \leq \kappa_{\rm G}\left(\|n_k\| + \tfrac{1}{2}\|n_k\|^2\right).$$

Using (2.3) and (3.43), we then obtain that

$$|\delta_k^{f,n}| \leq 2\kappa_{\rm G}\|n_k\|. \tag{3.46}$$

We now observe that (2.22), (3.42), and Lemma 3.3 imply that

$$\delta_k^{f,t} \geq \kappa_{\rm tC}\epsilon_f \min\left[\frac{\epsilon_f}{\kappa_{\rm G}}, \Delta_k\right] = \kappa_{\rm tC}\epsilon_f\Delta_k,$$

where we used (3.43) and (2.12) to deduce the last equality. Combining this bound with (3.46), we obtain that

$$\frac{|\delta_k^{f,n}|}{\delta_k^{f,t}} \le \frac{2\kappa_{\rm G}\|n_k\|}{\kappa_{\rm tC}\epsilon_f\Delta_k} \le \frac{2\kappa_{\rm G}\kappa_{\mathcal{CS}}}{(\kappa_{\mathcal{CS}}-1)\kappa_{\rm tC}\epsilon_f}\,\frac{\|n_k\|}{\|t_k\|} \le 1-\kappa_\delta,$$

where we used (3.45) to derive the second inequality and (3.44) to deduce the third. It is now easy to see that this last inequality implies (2.31). □

We now turn to the case where $\|t_k\|$ is relatively small compared to $\|n_k\|$.

Lemma 3.16 *Suppose that $k \in \mathcal{C}_t$, that $\epsilon > 0$ is given, that*

$$\|t_k\| \le \varsigma(\epsilon)\|n_k\| \tag{3.47}$$

and that

$$\|c_k\| \le \min\left[\kappa_{\rm c}, \frac{\kappa_{\Delta {\rm c}1}}{\kappa_{\Delta {\rm c}2}\kappa_{\rm J}}, \frac{\kappa_{\rm sRn}^2\kappa_{\rm R}(1-\eta_1)}{(1+\varsigma(\epsilon))^2[\kappa_{\theta 1}(1+\varsigma(\epsilon))\kappa_{\rm n}+\kappa_{\theta 2}]}\right]. \tag{3.48}$$

Then iteration k is successful, $\rho_k^c \ge \eta_1$ and $\Delta_{k+1}^c \ge \Delta_k^c$.

Proof. First observe that part (i) of Lemma 3.5 with the choice $\kappa_{\rm gen} = \varsigma(\epsilon)$ implies that $k \in \mathcal{A}$. Now assume that $k \notin \mathcal{S}$ and note that (3.48) then allows us to apply Lemmas 3.13 and 3.14. Since Lemma 2.1 ensures that the first part of (2.39) holds, we deduce that $\rho_k^c < \eta_1$ since the iteration is unsuccessful. But using now successively (2.39), (3.17), (3.36), (3.40), the triangle inequality, (3.47), (2.5) and (3.48), we have

$$\begin{aligned}
|\rho_k^c - 1| &= \left|\frac{\theta(x_k^+) - \frac{1}{2}\|c_k + J_k s_k\|^2}{\delta_k^c}\right| \\
&\le \left|\frac{\kappa_{\theta 1}\|s_k\|^3 + \kappa_{\theta 2}\|c_k\|\,\|s_k\|^2}{\kappa_{\rm R}\|s_k^R\|^2}\right| \\
&\le \frac{\|s_k\|^2}{\kappa_{\rm sRn}^2\kappa_{\rm R}\|n_k\|^2}\left[\kappa_{\theta 1}\|s_k\| + \kappa_{\theta 2}\|c_k\|\right] \\
&\le \frac{(1+\varsigma(\epsilon))^2}{\kappa_{\rm sRn}^2\kappa_{\rm R}}\left[\kappa_{\theta 1}(1+\varsigma(\epsilon))\|n_k\| + \kappa_{\theta 2}\|c_k\|\right] \\
&\le \frac{(1+\varsigma(\epsilon))^2}{\kappa_{\rm sRn}^2\kappa_{\rm R}}\left[\kappa_{\theta 1}(1+\varsigma(\epsilon))\kappa_{\rm n} + \kappa_{\theta 2}\right]\|c_k\| \\
&\le 1-\eta_1.
\end{aligned}$$

Thus $\rho_k^c \ge \eta_1$, which is a contradiction. As a consequence, iteration k is successful and the desired conclusion follows. □

We now return to the convergence properties of our algorithm, and, having covered in Lemma 3.12 the case of finitely many successful iterations, we consider the case where there are infinitely many of those. We start by assuming that they are all f-iterations for k large.

Lemma 3.17 *Suppose that $|\mathcal{S}| = +\infty$, that $|\mathcal{C}\cap\mathcal{S}| < +\infty$ and that no subsequence exists such that (3.12) holds. Then there exists an infinite subsequence indexed by $\mathcal{K}$ such that*

$$\lim_{k\to\infty, k\in\mathcal{K}} \|c_k\| = 0 \tag{3.49}$$

and

$$\lim_{k\to\infty, k\in\mathcal{K}} \pi_k = 0. \tag{3.50}$$

Proof. As a consequence of our assumptions, we immediately obtain that all successful iterations must belong to $\mathcal{F}$ for k sufficiently large, and that there are infinitely many of them. We also deduce that the sequence $\{f(x_k)\}$ is monotonically decreasing for large enough k. Assume now, for the purpose of deriving a contradiction, that (3.28) holds. Hence (3.29) also holds because of Lemma 3.10. Moreover, (2.22), Lemma 3.3 and (3.29) together give that, for all $k \in \mathcal{S}$ sufficiently large,

$$\delta_k^{f,t} \geq \kappa_{\rm tC}\epsilon_f \min\left[\frac{\epsilon_f}{\kappa_{\rm G}}, \min[\Delta_k^c, \epsilon_{\mathcal{F}}]\right]. \tag{3.51}$$

Assume now that there exists an infinite subsequence indexed by $\mathcal{K}_f \subseteq \mathcal{S}$ such that $\{\Delta_k^c\}$ converges to zero in $\mathcal{K}_f$. Since Δ_k^c is only decreased at unsuccessful c-iterations, this in turn implies that there is a subsequence of such iterations indexed $\mathcal{K}_c \subseteq \mathcal{C} \setminus \mathcal{S}$ with Δ_k^c converging to zero. Because of Lemma 3.7, we may also assume, without loss of generality, that $\mathcal{K}_c \subseteq \mathcal{C}_t \setminus \mathcal{S}$. Also note that, since $\theta_k^{\max}$ is only updated at successful c-iterations, the assumption that $|\,\mathcal{C} \cap \mathcal{S}| < +\infty$ ensures that

$$\theta_k^{\max} = \theta_\infty^{\max} \tag{3.52}$$

for some $\theta_\infty^{\max} > 0$ and for all k sufficiently large. Also observe that the second part of Lemma 3.9 implies that $\|J_k^T c_k\|$ converges to zero along $\mathcal{K}_c$. Because no subsequence exists such that (3.12) holds, we then obtain from (3.13) and (2.1) that $\|c_k\|$ and θ_k converge to zero along $\mathcal{K}_c$. Moreover, using the convergence of both Δ_k^c and $\|c_k\|$ to zero, the inclusion $\mathcal{K}_c \subseteq \mathcal{C}$ and Lemma 3.8, we deduce that $k \in \mathcal{A}$ for k sufficiently large. Using now Lemma 3.14 and the inclusion $\mathcal{K}_c \subseteq \mathcal{C} \setminus \mathcal{S}$, we therefore obtain that $\|c_k + J_k s_k\| \leq \kappa_{\rm cld}\|c_k\|^2$ for k sufficiently large. As a consequence, and taking (2.1), (3.52), (3.17) and Lemma 2.4 into account, we deduce that, for large enough k,

$$\theta(x_k^+) \leq \kappa_{\rm cld}\theta_\infty^{\max} + \kappa_{\theta 1}[\Delta_k^c]^3 + \kappa_{\theta 2}\sqrt{2\theta_\infty^{\max}}[\Delta_k^c]^2$$

and thus (2.33) holds as soon as

$$[\Delta_k^c]^2 \leq \min\left[1, \frac{(1-\kappa_{\rm cld})\theta_\infty^{\max}}{\kappa_{\theta 1} + \kappa_{\theta 2}\sqrt{2\theta_\infty^{\max}}}\right],$$

and therefore

$$\theta(x_k^+) \leq \theta_k^{\max} \tag{3.53}$$

for all $k \in \mathcal{K}_c$ sufficiently large because Δ_k^c converges to zero along that subsequence.

Now assume that $|\,\mathcal{A} \cap \mathcal{K}_c| < +\infty$. Not only (2.33) holds for $k \in \mathcal{K}_c$ large enough, but (2.31) also holds as soon as $n_k = 0$ because then $\delta_k^f = \delta_k^{f,t}$, and this is the case for all $k \in \mathcal{K}_c$ sufficiently large. Finally, $t_k \neq 0$ for these k since otherwise $k \in \mathcal{Y}$. But these arguments then imply that $k \in \mathcal{F}$, which is a contradiction. Thus $|\,\mathcal{A} \cap \mathcal{K}_c| = +\infty$ and we may restrict our attention to the subsequence indexed by $\mathcal{K}_1 = \mathcal{A} \cap \mathcal{K}_c$, yielding that $\mathcal{K}_1 \subseteq \mathcal{A}$.

Assume that

$$\|t_k\| \leq \varsigma(\epsilon_f)\|n_k\| \tag{3.54}$$

holds for all $k \in \mathcal{K}_1$ sufficiently large. We may then apply Lemma 3.16 and deduce that $\|c_k\|$ must be bounded away from zero along $\mathcal{K}_c$, which is impossible because we have already proved that $\|c_k\|$ converges to zero along this subsequence. Hence there exists an infinite subsequence indexed by $\mathcal{K}_2 \subseteq \mathcal{K}_1$ such that (3.54) fails (i.e. (3.44) holds) for $k \in \mathcal{K}_2$. We now verify that we may apply Lemma 3.15 for k sufficiently large in $\mathcal{K}_2 \subseteq \mathcal{C} \setminus \mathcal{S}$. First, (3.42) holds because we have assumed (3.28). The condition (3.43) also holds for sufficiently large k in $\mathcal{K}_2$ because we have Δ_k^c converges to zero along $\mathcal{K}_c$ and hence along $\mathcal{K}_2$. Applying Lemma 3.15, we then deduce that $t_k \neq 0$ and

(2.31) holds for all $k \in \mathcal{K}_2$ sufficiently large. But, in view of (3.53), this implies that $k \in \mathcal{F}$ for such k, which is impossible.

We therefore conclude that the sequence $\mathcal{K}_f$ described above cannot exist, and hence that there must exist an $\epsilon_* > 0$ such that $\Delta_k^c \geq \epsilon_*$ for $k \in \mathcal{S}$. Substituting this bound in (3.51) then yields that

$$\delta_k^{f,t} \geq \kappa_{\rm tC}\epsilon_f \min\left[\frac{\epsilon_f}{\kappa_{\rm G}}, \min[\epsilon_*, \epsilon_{\mathcal{F}}]\right] > 0 \tag{3.55}$$

for k sufficiently large, say $k \geq k_0$. But we also have that

$$f(x_{k_0}) - f(x_k) = \sum_{j=k_0, j\in\mathcal{S}}^{k-1} [f(x_j) - f(x_{j+1})] \geq \eta_1 \sum_{j=k_0, j\in\mathcal{S}}^{k-1} \delta_j^{f,t}. \tag{3.56}$$

This bound combined with (3.55) and the identity $|\mathcal{F} \cap \mathcal{S}| = +\infty$ then implies that f is unbounded below, which, in view of (2.46), contradicts (3.2). Hence (3.28) is impossible and we deduce that

$$\lim_{k\to\infty, k\in\mathcal{K}} \pi_k = 0 \tag{3.57}$$

for some index set $\mathcal{K} \subseteq \mathcal{F}$, which immediately gives (3.50). For all $k \in \mathcal{K} \subseteq \mathcal{F}$ we have by definition that $t_k \neq 0$ and thus (2.27) must hold. We then conclude from (3.57) that (3.49) must hold, which concludes the proof. □

After considering the case where the number of successful c-iterations is finite, we now turn to the situation where it is infinite. In the next two lemmas we first deduce global convergence for the problem of minimizing θ.

Lemma 3.18 *Suppose that $|\mathcal{C} \cap \mathcal{S}| = +\infty$. Then,*

$$\liminf_{k\to\infty, k\in\mathcal{C}\cap\mathcal{S}} \|J_k^T c_k\| = 0. \tag{3.58}$$

Proof. Assume, for the purpose of deriving a contradiction, that

$$\|J_k c_k\| \geq \epsilon_\theta \;\text{ for all }\; k \in \mathcal{C} \cap \mathcal{S} \tag{3.59}$$

and some $\epsilon_\theta > 0$. Observe that the value of $\theta_k^{\max}$ is updated (and reduced) in (2.41) at each of the infinitely many iterations indexed by $\mathcal{C} \cap \mathcal{S}$.

Let us first assume that the maximum in (2.41) is attained infinitely often by the first term. Since $\kappa_{\rm tx1} < 1$, we deduce that

$$\lim_{k\to\infty} \theta_k^{\max} = 0.$$

Using the uniform boundedness of the constraint Jacobian (3.1) and (2.46), we then immediately deduce from this limit that

$$\lim_{k\to\infty, k\in\mathcal{C}\cap\mathcal{S}} \|J_k^T c_k\| \leq \kappa_{\rm H} \lim_{k\to\infty, k\in\mathcal{C}\cap\mathcal{S}} \|c_k\| \leq \kappa_{\rm H} \lim_{k\to\infty, k\in\mathcal{C}\cap\mathcal{S}} \theta_k^{\max} = \kappa_{\rm H} \lim_{k\to\infty} \theta_k^{\max} = 0,$$

which is impossible in view of (3.59). Hence the maximum in (2.41) can only be attained a finite number of times by the first term. Now let $k \in \mathcal{C} \cap \mathcal{S}$ be the index of an iteration where the maximum is attained by the second term and observe that $k \in \mathcal{A}$ because of part (ii) of Lemma 3.5. Combining (2.46), (2.41), (2.39), (2.4), (3.3), Lemma 3.11 and (3.59), we obtain that

$$
\begin{aligned}
\theta_k^{\max} - \theta_{k+1}^{\max} &\geq \theta(x_k) - \theta_{k+1}^{\max} \\
&\geq (1-\kappa_{\text{tx2}})\big[\theta(x_k) - \theta(x_{k+1})\big] \\
&\geq (1-\kappa_{\text{tx2}})\eta_1 \delta_k^c \\
&\geq (1-\kappa_{\text{tx2}})\eta_1 \kappa_{\text{cn}} \delta_k^{c,n} \\
&\geq (1-\kappa_{\text{tx2}})\eta_1 \kappa_{\text{cn}} \kappa_{\text{nC}} \epsilon_\theta \min\left[\frac{\epsilon_\theta}{\kappa_{\text{H}}^2}, \min[\kappa_{\Delta\text{c3}}, \kappa_{\Delta\text{c4}}\epsilon_\theta]\right] \\
&> 0.
\end{aligned}
\tag{3.60}
$$

Since the value of $\theta_k^{\max}$ is monotonic, this last inequality and the infinite nature of $|\mathcal{C}\cap\mathcal{S}|$ implies that the sequence $\{\theta_k^{\max}\}$ is unbounded below, which obviously contradicts (2.46). Hence, the maximum in (2.41) also cannot be attained infinitely often by the second term. We must therefore conclude that our initial assumption (3.59) is impossible, which gives (3.58). □

Lemma 3.19 *Suppose that $|\mathcal{C}\cap\mathcal{S}| = +\infty$. Then either there is a subsequence indeed by $\mathcal{Z}$ such that (3.12) holds, or we have that*

$$\lim_{k\to\infty} \|c_k\| = 0, \tag{3.61}$$

$$\lim_{k\to\infty} \theta_k^{\max} = 0, \tag{3.62}$$

$$\lim_{k\to\infty} n_k = 0, \tag{3.63}$$

and

$$\lim_{k\to\infty} \delta_k^{f,n} = 0. \tag{3.64}$$

If, in addition to (3.61), (3.47) holds for some $\epsilon > 0$ and for all $k \in \mathcal{C}_t$, then there exists an $\epsilon_* > 0$ such that

$$\Delta_k^c \geq \epsilon_*, \tag{3.65}$$

for all k sufficiently large.

Proof. Assume that no $\mathcal{Z}$ exists such that (3.12) holds. Then Lemma 3.18, (2.1), and (3.1) imply that there must exist an infinite subsequence indexed by $\mathcal{G} \subseteq \mathcal{C}\cap\mathcal{S}$ such that

$$0 = \lim_{k\to\infty, k\in\mathcal{G}} \|J_k^T c_k\| = \lim_{k\to\infty, k\in\mathcal{G}} \|c_k\| = \lim_{k\to\infty, k\in\mathcal{G}} \theta(x_k). \tag{3.66}$$

As above, if the maximum in (2.41) is attained infinitely often in $\mathcal{G}$ by the first term, then we may conclude from the inequality $\kappa_{\text{tx1}} < 1$ and (2.41) that (3.62) must hold, and then (3.61) follows from (2.46) and (2.1). If this is not the case, we deduce from (2.41) that

$$\lim_{k\to\infty, k\in\mathcal{G}} \theta_{k+1}^{\max} \leq \lim_{k\to\infty, k\in\mathcal{G}} \theta(x_k) = 0.$$

and thus, because of the monotonicity of the sequence $\{\theta_k^{\max}\}$, that (3.62) and (3.61) again hold. The limit (3.61) and (2.5) then give that (3.63) holds, while (3.64) follows from the identity

$$-\delta_k^{f,n} = \langle g_k, n_k\rangle + \tfrac{1}{2}\langle n_k, G_k n_k\rangle, \tag{3.67}$$

the Cauchy-Schwarz inequality, (3.63), Lemma 3.3 and (3.1).

Suppose now that (3.47) holds for all $k \in \mathcal{C}_t$. It then follows from part (i) of Lemma 3.5 with the choice $\kappa_{\text{gen}} = \varsigma(\epsilon)$ that $\mathcal{C}_t \subseteq \mathcal{A}$. Consider k large enough to ensure (3.48), which is possible because of (3.61). Lemma 3.16 then implies that $\Delta_{k+1}^c \geq \Delta_k^c$ for all $k \in \mathcal{C}_t$ sufficiently large. In addition, Lemma 3.7 ensures that Δ_k^c is bounded below by a constant for all $k \in \mathcal{C}_w = \mathcal{C}\setminus\mathcal{C}_t$. These two observations and the fact that Δ_k^c is only decreased for $k \in \mathcal{C}$ finally imply (3.65). □

Observe that it is not crucial that $\theta_k^{\max}$ is updated at every iteration in $\mathcal{C} \cap \mathcal{S}$, but rather that such updates occur infinitely often in a subset of this set along which $\|J_k^T c_k\|$ converges to zero. Other mechanisms to guarantee this property are possible, such as updating $\theta_k^{\max}$ every p iteration in $\mathcal{C}\cap\mathcal{S}$ at which $\|J_k^T c_k\|$ decreases. Relaxed scheme of this type may have the advantage of not pushing $\theta_k^{\max}$ too quickly to zero, therefore allowing more freedom for f-iterations.
We next consider what can happen at unsuccessful c-iterations whenever $\|c_k\|$ and $\theta_k^{\max}$ are both small.

Lemma 3.20 *Suppose that $k \in \mathcal{C} \setminus \mathcal{S}$, that (3.38) holds, that*

$$\theta_k^{\max} \le \min\left[\left\{\min\left[1, \frac{1-\max[\kappa_{\rm cld}, \kappa_{\rm tt}, \kappa_{\theta\theta}]}{\kappa_{\theta 1} + \kappa_{\theta 2}\sqrt{2}}\right]\right\}^4, \kappa_{\mathcal{C}}^{\frac{12}{5}}\right] \tag{3.68}$$

and

$$\Delta_k^c \le [\theta_k^{\max}]^{\frac{5}{12}}. \tag{3.69}$$

Then (2.33) holds.

Proof. Observe first that (3.69) and (3.68) imply that $\Delta_k^c \le \kappa_{\rm C}$ and hence, in view of Lemma 3.7 that $k \in \mathcal{C}_t$.

First, assume that $k \in \mathcal{A}$. From (3.17), Lemma 3.14 (which is applicable because of (3.38)) and Lemma 2.4, we deduce that

$$\theta(x_k^+) \le \tfrac{1}{2}\kappa_{\rm cld}\|c_k\|^2 + \kappa_{\theta 1}[\Delta_k^c]^3 + \kappa_{\theta 2}\sqrt{2\theta_k^{\max}}[\Delta_k^c]^2.$$

Using (2.1), (2.46), (3.69) and (3.68) successively yields that

$$\theta(x_k^+) \le \theta_k^{\max}\left[\kappa_{\rm cld} + \kappa_{\theta 1}[\theta_k^{\max}]^{\frac{3}{12}} + \kappa_{\theta 2}\sqrt{2}\,[\theta_k^{\max}]^{\frac{4}{12}}\right] \le \theta_k^{\max},$$

which is (2.33). If $k \notin \mathcal{A}$, the successive use of (3.17), Lemma 2.4, (2.25), (2.26), (2.28), (3.69) and (3.68) then similarly gives that

$$\begin{aligned}
\theta(x_k^+) &\le \tfrac{1}{2}\|c_k + J_k^T s_k\|^2 + \kappa_{\theta 1}[\Delta_k^c]^3 + \kappa_{\theta 2}\sqrt{2\theta_k^{\max}}[\Delta_k^c]^2 \\
&\le \max[\kappa_{\rm tt}\theta_k^{\max}, \tfrac{1}{2}\vartheta_k] + \kappa_{\theta 1}[\Delta_k^c]^3 + \kappa_{\theta 2}\sqrt{2\theta_k^{\max}}[\Delta_k^c]^2 \\
&\le \max[\kappa_{\rm tt}\theta_k^{\max}, \theta_k] + \kappa_{\theta 1}[\Delta_k^c]^3 + \kappa_{\theta 2}\sqrt{2\theta_k^{\max}}[\Delta_k^c]^2 \\
&\le \max[\kappa_{\rm tt}, \kappa_{\theta\theta}]\theta_k^{\max} + \kappa_{\theta 1}[\Delta_k^c]^3 + \kappa_{\theta 2}\sqrt{2\theta_k^{\max}}[\Delta_k^c]^2 \\
&\le \theta_k^{\max}\left[\max[\kappa_{\rm tt}, \kappa_{\theta\theta}] + \kappa_{\theta 1}[\theta_k^{\max}]^{\frac{3}{12}} + \kappa_{\theta 2}\sqrt{2}\,[\theta_k^{\max}]^{\frac{4}{12}}\right] \\
&\le \theta_k^{\max}
\end{aligned}$$

which is (2.33). □

Convergence of the criticality measure π_k to zero then follows for a subsequence of iterations, as we now prove.

Lemma 3.21 *Suppose that $|\mathcal{C} \cap \mathcal{S}| = +\infty$. Then either there is a subsequence indexed by $\mathcal{Z}$ such that (3.12) holds, or (3.61) holds and*

$$\liminf_{k\to\infty} \pi_k = 0. \tag{3.70}$$

Proof. Assume that no subsequence exists such that (3.12) holds. We may then apply Lemma 3.19 and deduce that (3.61)–(3.64) hold. Assume now, again for the purpose of deriving a contradiction, that

$$\pi_k \ge \epsilon_f > 0 \tag{3.71}$$

for all $k \geq 0$. This assumption and Lemma 3.10 then guarantee that (3.29) holds for all k sufficiently large. Also, given $\epsilon = \epsilon_f$, it follows from (3.61) and (3.62) that there exists a $k_0 \in \mathcal{C}$ sufficiently large to ensure that (3.48) (and thus (3.34) and (3.38)) and (3.68) hold for $k \in \mathcal{C}$, $k \geq k_0$.

The first step of our proof is to show that, under these conditions, the trust-region radius Δ_k cannot be arbitrarily small compared to $\theta_k^{\max}$ along $\mathcal{C}$. To this aim, we distinguish two cases. The first (i) is when (3.47) holds for all $k \in \mathcal{C}_t$, $k \geq k_0$ sufficiently large. We may then apply Lemma 3.19 and deduce that (3.65) holds for all $k \in \mathcal{C}$ sufficiently large.

The second case (ii) is when the subsequence indexed by

$$\mathcal{K}_1 \stackrel{\text{def}}{=} \{k \in \mathcal{C}_t \mid \|t_k\| \geq \varsigma(\epsilon_f)\|n_k\|\}$$

is infinite.

Suppose first, in this case, that there is a $k \in (\mathcal{K}_1 \setminus \mathcal{S}) \setminus \mathcal{A}$ with $k \geq k_0$. Then t_k must be computed (otherwise $k \in \mathcal{Y}$) and (2.31) holds since $n_k = 0$. But Lemma 3.20 then gives that, if (3.69) holds, then (2.33) also holds and thus $k \in \mathcal{F}$, which is impossible. Hence (3.69) must fail and

$$\Delta_k^c > [\theta_k^{\max}]^{\frac{5}{12}} \quad \text{for all } \; k \in (\mathcal{K}_1 \setminus \mathcal{S}) \setminus \mathcal{A}, \; k \geq k_0. \tag{3.72}$$

Consider now a $k \in (\mathcal{K}_1 \setminus \mathcal{S}) \cap \mathcal{A}$ such that $k \geq k_0$ and note that the definition of $\mathcal{K}_1$ ensures that (3.44) holds with $\epsilon = \epsilon_f$. Suppose that (3.43) and (3.69) both hold. Then, on one hand, the first of these inequalities and Lemma 3.15 imply that $t_k \neq 0$ and that (2.31) holds. On the other hand, (3.69) and Lemma 3.20 ensure that (2.33) also holds. As consequence, $k \in \mathcal{F}$, which is again impossible. We therefore deduce that one of (3.43) and (3.69) must fail, yielding that

$$\Delta_k^c \geq \min\left[\kappa_{\delta\mathcal{C}}, [\theta_k^{\max}]^{\frac{5}{12}}\right] \quad \text{for all } \; k \in (\mathcal{K}_1 \setminus \mathcal{S}) \cap \mathcal{A}, \; k \geq k_0.$$

Combining (3.72) and this inequality, we obtain that

$$\Delta_k^c \geq \min\left[\kappa_{\delta\mathcal{C}}, [\theta_k^{\max}]^{\frac{5}{12}}\right] \quad \text{for all } \; k \in \mathcal{K}_1 \setminus \mathcal{S}, \; k \geq k_0. \tag{3.73}$$

Moreover Lemma 3.16 implies that any $k \in \mathcal{C}_t \setminus \mathcal{K}_1$, $k \geq k_0$, must belong to $\mathcal{S}$, and hence that $\mathcal{C}_t \setminus \mathcal{S} = \mathcal{K}_1 \setminus \mathcal{S}$ beyond k_0. Thus (3.73) holds for all $k \in \mathcal{C}_t \setminus \mathcal{S}$, $k \geq k_0$. Using this conclusion and Lemma 3.7, we therefore deduce that

$$\Delta_k^c \geq \min\left[\kappa_{\delta\mathcal{C}}, \kappa_{\mathcal{C}}, [\theta_k^{\max}]^{\frac{5}{12}}\right]$$

for all $k \in \mathcal{C} \setminus \mathcal{S}$ sufficiently large. But we know that Δ_k^c is only decreased at unsuccessful c-iterations (at which $\theta_k^{\max}$ is unchanged), while it is only increased at successful c-iterations (at which $\theta_k^{\max}$ is decreased) or at successful f-iterations (at which $\theta_k^{\max}$ is unchanged). Thus we obtain that, in case (ii),

$$\Delta_k^c \geq \gamma_1 \min\left[\kappa_{\delta\mathcal{C}}, \kappa_{\mathcal{C}}, [\theta_k^{\max}]^{\frac{5}{12}}\right]$$

for all k sufficiently large. Considering the two cases distinguished above together then gives that, for $k \in \mathcal{C}$ large enough,

$$\Delta_k^c \geq \min\left[\epsilon_*, \gamma_1\kappa_{\delta\mathcal{C}}, \gamma_1\kappa_{\mathcal{C}}, \gamma_1[\theta_k^{\max}]^{\frac{5}{12}}\right],$$

and therefore, taking (2.12) and (3.29) into account, that, for $k \in \mathcal{C}$ large enough,

$$\Delta_k \geq \min\left[\epsilon_{\mathcal{F}}, \epsilon_*, \gamma_1\kappa_{\delta\mathcal{C}}, \gamma_1\kappa_{\mathcal{C}}, \gamma_1[\theta_k^{\max}]^{\frac{5}{12}}\right] \stackrel{\text{def}}{=} \min\left[\kappa_{\Delta\infty}, \gamma_1[\theta_k^{\max}]^{\frac{5}{12}}\right]. \tag{3.74}$$

The second step in our proof is to deduce the contradiction sought from (3.71). We start by observing that, if iteration k is a successful c-iteration, then (2.33) must hold because of (2.46) and the second part of (2.39). The successful c-iterations thus asymptotically come in two types:

1. iterations for which the tangential step has been computed but (2.31) fails,
2. iterations for which the tangential step has not been computed.

Assume first that there is an infinite number of successful c-iterations of type 1. Since (2.31) does not hold, we deduce that, for the relevant indices k,

$$\frac{|\delta_k^{f,n}|}{\delta_k^{f,t}} > 1 - \kappa_\delta. \tag{3.75}$$

But, as in the proof of Lemma 3.15, we deduce from (2.5) and (3.61) that

$$|\delta_k^{f,n}| \le \kappa_{\rm G}\|n_k\| + \tfrac{1}{2}\kappa_{\rm G}\|n_k\|^2 \le \kappa_{\rm n}\kappa_{\rm G}\|c_k\|\,(1+\tfrac{1}{2}\kappa_{\rm n}\|c_k\|) \le 2\kappa_{\rm n}\kappa_{\rm G}\|c_k\|$$

for large enough k. Moreover, using (2.22), (3.71), (3.74), (3.62) and (2.46), we verify that

$$\begin{aligned}
\delta_k^{f,t} &\ge \kappa_{\rm tC}\epsilon_f \min\left[\frac{\epsilon_f}{\kappa_{\rm G}}, \Delta_k\right]\\
&\ge \kappa_{\rm tC}\epsilon_f \min\left[\frac{\epsilon_f}{\kappa_{\rm G}}, \kappa_{\Delta\infty}, \gamma_1[\theta_k^{\max}]^{\frac{5}{12}}\right]\\
&= \kappa_{\rm tC}\epsilon_f\gamma_1[\theta_k^{\max}]^{\frac{5}{12}}\\
&\ge \kappa_{\rm tC}\epsilon_f\gamma_1[\tfrac{1}{2}]^{\frac{5}{12}}\|c_k\|^{\frac{5}{6}}\\
&\ge \tfrac{1}{2}\kappa_{\rm tC}\epsilon_f\gamma_1\|c_k\|^{\frac{5}{6}}
\end{aligned}$$

for large enough k. Combining these latter two inequalities and using (3.61) then yields that

$$\frac{|\delta_k^{f,n}|}{\delta_k^{f,t}} \le \frac{4\kappa_{\rm n}\kappa_{\rm G}\|c_k\|^{\frac{1}{6}}}{\kappa_{\rm tC}\epsilon_f\gamma_1} \le 1-\kappa_\delta$$

for k large enough. But this last inequality contradicts (3.75). Hence this situation is impossible.

Assume otherwise that there is an infinite number of successful c-iterations of type 2. These iterations occur because either (2.13) or (2.27) fails, the latter being impossible since both (3.71) and (3.61) hold. But the fact that $t_k = 0$ implies that $n_k \ne 0$ (since otherwise $k \in \mathcal{Y}$). Moreover, using (2.5), (3.74), (3.62), (2.46) and (3.61), we see that

$$\frac{\|n_k\|}{\Delta_k} \le \frac{\kappa_{\rm n}\|c_k\|}{\gamma_1[\theta_k^{\max}]^{\frac{5}{12}}} \le \frac{\kappa_{\rm n}\|c_k\|}{\gamma_1[\frac{1}{2}\|c_k\|^2]^{\frac{5}{12}}} \le \kappa_{\rm B}$$

for k large enough, which yields that (2.13) holds. We may therefore conclude that an impossible situation occurs for infinite subsequences of each of the two types of successful c-iterations. This in turn implies that $|\mathcal{C}\cap\mathcal{S}|$ is finite, which is also a contradiction. Our assumption (3.71) is therefore impossible, and (3.70) follows. □

We now combine our results so far and state a first important convergence property of our algorithm.

Theorem 3.22 *As long as infeasible stationary points are avoided, there exists a subsequence indexed by $\mathcal{K}$ such that (3.5), (3.7) and (3.8) hold, and thus at least one limit point of the sequence $\{x_k\}$ (if any) is first-order critical. Moreover, we also have that (3.61) holds when $|\mathcal{C}\cap\mathcal{S}| = +\infty$.*

Proof. The desired conclusions immediately follow from Lemmas 3.2, 3.12, 3.17, 3.19 and 3.21. □

3.3 A true "limit" result

Our intention is now to prove that the complete sequences $\{\pi_k\}$ and $\{\|P_k g_k\|\}$ both converge to zero, rather than merely subsequences. The first step to achieve this objective is to prove that the projection $P(x)$ onto the nullspace of the Jacobian $J(x)$ is Lipschitz continuous when x is sufficiently close to the feasible domain.

Lemma 3.23 *There exists a constant $\kappa_{\rm P} > 0$ such that, for all x_1 and x_2 satisfying $\max\left[\|c(x_1)\|, \|c(x_2)\|\right] \leq \kappa_{\rm c}$, we have that*

$$\|P(x_1) - P(x_2)\| \leq \kappa_{\rm P}\|x_1 - x_2\|. \tag{3.76}$$

Proof. Because of (3.13) and our assumption on $c(x_1)$ and $c(x_2)$, we know that

$$P(x_i) = I - J(x_i)^T[J(x_i)J(x_i)^T]^{-1}J(x_i) \quad (i = 1, 2). \tag{3.77}$$

Denoting $J_1 \stackrel{\rm def}{=} J(x_1)$ and $J_2 \stackrel{\rm def}{=} J(x_2)$, we first observe that

$$[J_1J_1^T]^{-1} - [J_2J_2^T]^{-1} = [J_1J_1^T]^{-1}\big((J_2 - J_1)J_1^T - J_2(J_1 - J_2)^T\big)[J_2J_2^T]^{-1}. \tag{3.78}$$

But the mean-value theorem and (3.1) imply that, for $i = 1, \ldots, m$,

$$\begin{aligned}
\|\nabla_x c_i(x_1) - \nabla_x c_i(x_2)\| &\leq \left\|\int_0^1 \nabla_{xx}c_i(x_1 + t(x_2 - x_1))(x_1 - x_2)\,dt\right\| \\
&\leq \max_{t\in[0,1]} \|\nabla_{xx}c_i(x_1 + t(x_2 - x_1))\|\,\|x_1 - x_2\| \\
&\leq \kappa_{\rm H}\|x_1 - x_2\|,
\end{aligned}$$

which in turn yields that

$$\|(J_1 - J_2)^T\| = \|J_1 - J_2\| \leq m\kappa_{\rm H}\|x_1 - x_2\|. \tag{3.79}$$

Hence, using (3.78), (3.1) and (3.13), we obtain that

$$\|[J_1J_1^T]^{-1} - [J_2J_2^T]^{-1}\| \leq \frac{2m\kappa_{\rm H}^2}{\kappa_{\rm J}^4}\|x_1 - x_2\|. \tag{3.80}$$

Computing now the difference between $P(x_1)$ and $P(x_2)$ and using (3.77), we deduce that

$$\begin{aligned}
P(x_1) - P(x_2) &= J_1^T[J_1J_1^T]^{-1}(J_2 - J_1) + (J_2 - J_1)^T[J_2J_2^T]^{-1}J_2 \\
&\quad - J_1^T\left([J_1J_1^T]^{-1} - [J_2J_2^T]^{-1}\right)J_2
\end{aligned}$$

and thus, using (3.1) and (3.13) again with (3.79) and (3.80),

$$\|P(x_1) - P(x_2)\| \leq \frac{m\kappa_{\rm H}^2}{\kappa_{\rm J}^2}\|x_1 - x_2\| + \frac{m\kappa_{\rm H}^2}{\kappa_{\rm J}^2}\|x_1 - x_2\| + \frac{2m\kappa_{\rm H}^4}{\kappa_{\rm J}^4}\|x_1 - x_2\|.$$

This then yields (3.76) with $\kappa_{\rm L} = \frac{2m\kappa_{\rm H}^2}{\kappa_{\rm J}^2}\left(1 + \frac{\kappa_{\rm H}^2}{\kappa_{\rm J}^2}\right)$. □

We now refine our interpretation of the criticality measure π_k, and verify that it approximates the norm of the projected gradient when the constraint violation is small enough.

Lemma 3.24 *Suppose that*

$$\min\left[\tfrac{1}{2}\|P_kg_k\|, \tfrac{1}{12}\|P_kg_k\|^2\right] > \kappa_{\rm H}\kappa_{\rm G}\kappa_{\rm n}\|c_k\| + \kappa_{\rm H}^2\omega_y(\|c_k\|). \tag{3.81}$$

Then we have that

$$\pi_k = \psi_k\|P_kg_k\| \tag{3.82}$$

for some $\psi_k \in [\frac{1}{9}, \frac{11}{3}]$.

Proof. From (2.14) we know that

$$y_k = -[J_k^T]^I g_k^{\mathrm{N}} + \omega_y(\|c_k\|) u_k$$

for some u_k with $\|u_k\| \le 1$. Therefore, using (2.17) and (2.7) yields that

$$r_k = (I - J_k^T [J_k^T]^I) g_k^{\mathrm{N}} + \omega_y(\|c_k\|) J_k^T u_k = P_k(g_k + G_k n_k) + \omega_y(\|c_k\|) J_k^T u_k \quad (3.83)$$

and thus, using (2.21),

$$\pi_k \left(\|P_k g_k + P_k G_k n_k + \omega_y(\|c_k\|) J_k^T u_k\| \right) = \pi_k \|r_k\| = \langle g_k, r_k \rangle + \langle G_k n_k, r_k \rangle. \quad (3.84)$$

Now, using the triangle inequality, (3.1), (2.5), (3.81), Lemma 3.3 and the bounds $\|P_k\| \le 1$, $\|u_k\| \le 1$, and $\kappa_{\mathrm{H}} \ge 1$, we verify that

$$\begin{aligned} \|r_k\| &= \|P_k g_k + P_k G_k n_k + \omega_y(\|c_k\|) J_k^T u_k\| \\ &\le \|P_k g_k\| + \|P_k G_k n_k\| + \omega_y(\|c_k\|) \|J_k^T u_k\| \\ &\le \|P_k g_k\| + \|G_k n_k\| + \kappa_{\mathrm{H}} \omega_y(\|c_k\|) \|u_k\| \\ &\le \|P_k g_k\| + \kappa_{\mathrm{G}} \kappa_{\mathrm{n}} \|c_k\| + \kappa_{\mathrm{H}} \omega_y(\|c_k\|) \\ &< \|P_k g_k\| + \tfrac{1}{2} \|P_k g_k\|. \end{aligned}$$

Similarly,

$$\|r_k\| \ge \|P_k g_k\| - \|P_k G_k n_k + \omega_y(\|c_k\|) J_k^T u_k\| > \|P_k g_k\| - \tfrac{1}{2} \|P_k g_k\|.$$

Thus $\|r_k\| = \|P_k g_k\|(1 + \alpha_k)$ for some $|\alpha_k| < \frac{1}{2}$. Substituting this relation in (3.84) and using (3.83) and the symmetric and idempotent nature of the orthogonal projection P_k, we obtain that

$$\pi_k = \frac{1}{1 + \alpha_k} \|P_k g_k\| + \frac{\langle g_k, P_k G_k n_k + \omega_y(\|c_k\|) J_k^T u_k \rangle}{(1 + \alpha_k) \|P_k g_k\|} + \frac{\langle G_k n_k, r_k \rangle}{\|r_k\|}.$$

But the Cauchy-Schwarz inequality, (2.5), (3.1), Lemma 3.3, the bounds $\|P_k\| \le 1$ and $\kappa_{\mathrm{H}} \ge 1$ and (3.81) then ensure that

$$\left| \frac{\langle G_k n_k, r_k \rangle}{\|r_k\|} \right| \le \kappa_{\mathrm{G}} \kappa_{\mathrm{n}} \|c_k\| < \tfrac{1}{2} \|P_k g_k\|$$

and that

$$\left| \frac{\langle g_k, P_k G_k n_k + \omega_y(\|c_k\|) J_k^T u_k \rangle}{(1 + \alpha_k) \|P_k g_k\|} \right| \le \frac{\kappa_{\mathrm{H}} \kappa_{\mathrm{G}} \kappa_{\mathrm{n}} \|c_k\| + \kappa_{\mathrm{H}}^2 \omega_y(\|c_k\|)}{(1 + \alpha_k) \|P_k g_k\|} < \frac{1}{12(1 + \alpha_k)} \|P_k g_k\|.$$

Hence we deduce that, for some $\beta_k \in [-\frac{1}{2}, \frac{1}{2}]$ and some $\zeta_k \in [-\frac{1}{12}, \frac{1}{12}]$,

$$\pi_k = \frac{1 + \zeta_k}{1 + \alpha_k} \|P_k g_k\| + \beta_k \|P_k g_k\| = \frac{1 + \zeta_k + \beta_k + \alpha_k \beta_k}{1 + \alpha_k} \|P_k g_k\|.$$

This in turn yields (3.82) because

$$\psi_k \stackrel{\mathrm{def}}{=} \frac{1 + \zeta_k + \beta_k + \alpha_k \beta_k}{1 + \alpha_k} \in [\tfrac{1}{9}, \tfrac{11}{3}]$$

for all $(\alpha_k, \beta_k, \zeta_k) \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{12}, \frac{1}{12}]$. □

The preceding result ensures the following simple but useful technical consequence.

Lemma 3.25 *Suppose that* $\epsilon > 0$ *is given and that*

$$\kappa_{\rm H}\kappa_{\rm G}\kappa_{\rm n}\|c_k\| + \kappa_{\rm H}^2\omega_y(\|c_k\|) \leq \epsilon. \tag{3.85}$$

Then, for any $\alpha > \frac{1}{5}$,

$$\min\left[\tfrac{1}{2}\|P_k g_k\|, \tfrac{1}{12}\|P_k g_k\|^2\right] \geq 5\alpha\epsilon \quad \textit{implies that} \quad \pi_k \geq \alpha\epsilon.$$

Proof. Assume first that (3.81) fails. We then obtain, using (3.85), that

$$5\alpha\epsilon \leq \min\left[\tfrac{1}{2}\|P_k g_k\|, \tfrac{1}{12}\|P_k g_k\|^2\right] \leq \kappa_{\rm H}\kappa_{\rm G}\kappa_{\rm n}\|c_k\| + \kappa_{\rm H}^2\omega_y(\|c_k\|) \leq \epsilon,$$

which is impossible because $\alpha > \frac{1}{5}$. Hence (3.81) must hold. In this case, we see, using Lemma 3.24, that

$$\tfrac{1}{2}\pi_k = \tfrac{1}{2}\psi_k\|P_k g_k\| \geq \psi_k \min\left[\tfrac{1}{2}\|P_k g_k\|, \tfrac{1}{12}\|P_k g_k\|^2\right] \geq \tfrac{5}{9}\alpha\epsilon > \tfrac{1}{2}\alpha\epsilon,$$

as desired. □

We now examine the consequences of the existence of a subsequence of consecutive f-iterations where π_k is bounded away from zero.

Lemma 3.26 *Suppose that there exist* $k_1 \in \mathcal{S}$ *and* $k_2 \in \mathcal{S}$ *with* $k_2 > k_1$ *such that all successful iterations between* k_1 *and* $k_2 - 1$ *are* f*-iterations, i.e.*

$$\{k_1, \ldots, k_2 - 1\} \cap \mathcal{S} \subseteq \mathcal{F}, \tag{3.86}$$

with the property that

$$\pi_j \geq \epsilon \quad \textit{for all} \quad j \in \{k_1, \ldots, k_2 - 1\} \cap \mathcal{S} \tag{3.87}$$

for some $\epsilon > 0$. *Assume furthermore that*

$$f(x_{k_1}) - f(x_{k_2}) \leq \frac{\eta_1\kappa_\delta\kappa_{\rm tC}\epsilon^2}{2\kappa_{\rm G}}. \tag{3.88}$$

Then

$$\|x_{k_1} - x_{k_2}\| \leq \frac{1}{\eta_1\kappa_\delta\kappa_{\rm tC}\epsilon}\left[f(x_{k_1}) - f(x_{k_2})\right]. \tag{3.89}$$

Proof. Consider a successful iteration j in the range $k_1, \ldots, k_2 - 1$ and note that the sequence $\{f(x_j)\}_{j=k_1}^{k_2}$ is monotonically decreasing. We then deduce from (2.22), (3.87), and Lemma 3.3 that

$$\delta_j^{f,t} \geq \kappa_{\rm tC}\pi_j \min\left[\frac{\pi_j}{1 + \|G_j\|}, \Delta_j\right] \geq \kappa_{\rm tC}\epsilon \min\left[\frac{\epsilon}{\kappa_{\rm G}}, \Delta_j\right].$$

Since $j \in \mathcal{S}$, we may use the previous bound, (2.35), and (2.31) to conclude that

$$f(x_j) - f(x_{j+1}) \geq \eta_1\delta_k^f \geq \eta_1\kappa_\delta\kappa_{\rm tC}\epsilon \min\left[\frac{\epsilon}{\kappa_{\rm G}}, \Delta_j\right]. \tag{3.90}$$

But the bound (3.88) and the inequality $f(x_j) - f(x_{j+1}) \leq f(x_{k_1}) - f(x_{k_2})$ yield together that the minimum in the right-hand side of (3.90) must be achieved by the second term. This in turn implies that

$$\|x_j - x_{j+1}\| \leq \Delta_j \leq \frac{1}{\eta_1\kappa_\delta\kappa_{\rm tC}\epsilon}\left[f(x_j) - f(x_{j+1})\right],$$

where we have used (2.47) to derive the first inequality. Summing now over all successful iterations from k_1 to $k_2 - 1$ and using the triangle inequality, we therefore obtain that

$$\|x_{k_1} - x_{k_2}\| \leq \sum_{j=k_1, j\in\mathcal{S}}^{k_2-1} \|x_j - x_{j+1}\| \leq \frac{1}{\eta_1 \kappa_\delta \kappa_{\rm tC} \epsilon} \sum_{j=k_1, j\in\mathcal{S}}^{k_2-1} \big[f(x_j) - f(x_{j+1})\big]$$

and (3.89) follows. □

We now extend Lemma 3.17 by showing that the constraint violation goes to zero not only along the subsequence for which the criticality π_k goes to zero, but actually along the complete sequence of iterates.

Lemma 3.27 *Suppose that* $|\,\mathcal{C}\cap\mathcal{S}\,| < +\infty$, *that* $|\,\mathcal{S}\,| = +\infty$, *that no subsequence exists such that (3.12) holds, and that* ω_t *is strictly increasing on* $[0, t_\omega]$ *for some* $t_\omega > 0$. *Then*

$$\lim_{k\to\infty} \|c_k\| = 0.$$

Proof. Let k_0 be the index of the last successful iteration in $\mathcal{C}$ (or -1 if there is none). Thus all successful iterations beyond k_0 must be f-iterations. In this case, we know that the sequence $\{f(x_k)\}$ is monotonically decreasing (by the mechanism of the algorithm) and bounded below by $f_{\rm low}$ because of (3.2); it is thus convergent to some limit $f_* \geq f_{\rm low}$. Assume first that there exists a subsequence indexed by $\mathcal{K}_c \subseteq \mathcal{F}\cap\mathcal{S}$ such that

$$\|c_k\| \geq \epsilon_0$$

for some $\epsilon_0 > 0$ and all $k \in \mathcal{K}_c$ with $k > k_0$. Because of (2.27) and the monotonicity of ω_t, we then deduce that

$$\pi_k \geq \omega_t(\epsilon_0)$$

for all $k \in \mathcal{K}_c$ with $k > k_0$. On the other hand, Lemma 3.17 implies the existence of an infinite subsequence $\mathcal{K}$ such that (3.5) and (3.7) both hold. We now choose an $\epsilon > 0$ small enough to ensure that

$$\epsilon \leq \min\left[\tfrac{1}{2}\omega_t(\epsilon_0), t_\omega\right] \quad \text{and} \quad \omega_t^{-1}(\epsilon) + \tfrac{1}{4}\epsilon \leq \tfrac{1}{2}\epsilon_0. \tag{3.91}$$

(Note that the first part of the condition and our assumption on ω_t ensures that this bounding function is invertible for all t sufficiently small.) We next choose an index $k_1 \in \mathcal{K}_c$ large enough to ensure that $k_1 > k_0$ and also that

$$f_{k_1} - f_* \leq \min\left[\frac{\eta_1\kappa_\delta\kappa_{\rm tC}\epsilon^2}{2\kappa_{\rm G}}, \frac{\eta_1\kappa_\delta\kappa_{\rm tC}\epsilon^2}{4\kappa_{\rm H}}\right], \tag{3.92}$$

which is possible since $\{f(x_k)\}$ converges in a monotonically decreasing manner to f_*. We finally select k_2 to be the first index in $\mathcal{K}$ after k_1 such that

$$\pi_j \geq \epsilon \quad \text{for all} \quad k_1 \leq j < k_2, j \in \mathcal{S}, \quad \text{and} \quad \pi_{k_2} < \epsilon. \tag{3.93}$$

Because $f(x_{k_1}) - f(x_{k_2}) \leq f(x_{k_1}) - f_*$ and (3.92), we may then apply Lemma 3.26 to the iterations k_1 and k_2, and deduce that (3.89) holds, and therefore, using (3.88) we obtain that

$$\|x_{k_1} - x_{k_2}\| \leq \frac{\epsilon}{4\kappa_{\rm H}}.$$

Thus, using the vector-valued mean-value theorem, we then obtain that

$$\begin{aligned}
\|c_{k_1} - c_{k_2}\| &\leq \left\|\int_0^1 J(x_{k_1} + t(x_{k_2} - x_{k_1}))(x_{k_1} - x_{k_2})\,dt\right\| \\
&\leq \max_{t\in[0,1]} \|J(x_{k_1} + t(x_{k_2} - x_{k_1}))\|\,\|x_{k_1} - x_{k_2}\| \\
&\leq \kappa_{\rm H}\|x_{k_1} - x_{k_2}\| \\
&\leq \tfrac{1}{4}\epsilon.
\end{aligned}$$

As a consequence, using the triangle inequality, the fact that $\omega_t(\|c_{k_2}\|) \le \pi_{k_2}$ (since $k_2 \in \mathcal{F}$), (3.93), and the second part of (3.91), we deduce that

$$\epsilon_0 \le \|c_{k_1}\| \le \|c_{k_2}\| + \tfrac{1}{4}\epsilon \le \omega_t^{-1}(\pi_{k_2}) + \tfrac{1}{4}\epsilon \le \omega_t^{-1}(\epsilon) + \tfrac{1}{4}\epsilon \le \tfrac{1}{2}\epsilon_0$$

which is a contradiction. Hence our initial assumption on the existence of the subsequence $\mathcal{K}_c$ is impossible and $\|c_k\|$ must converge to zero, as required. □

Our next result analyzes some technical consequences of the fact that there might be an infinite number of c-iterations. In particular, it indicates that feasibility improves linearly at c-iterations for sufficiently large k, and hence that these iterations must play a diminishing role as k increases.

Lemma 3.28 *Suppose that $|\mathcal{C} \cap \mathcal{S}| = +\infty$ and that no subsequence exists such that (3.12) holds. Then both $\{\theta_k\}$ and $\{\theta_k^{\max}\}$ converge linearly to zero along $\mathcal{C} \cap \mathcal{S}$, i.e. there exist $\kappa_\theta \in (0,1)$ and $\kappa_{\theta m} \in (0,1)$ such that, for $k \in \mathcal{C} \cap \mathcal{S}$ sufficiently large,*

$$\theta_{k+1} < \kappa_\theta \theta_k \tag{3.94}$$

and

$$\theta_{k+1}^{\max} \le \kappa_{\theta m} \theta_k^{\max}. \tag{3.95}$$

Proof. We first note that (3.61) holds because of Lemma 3.19, which implies that (3.13) also holds for k sufficiently large.

Now let k_c be the index of the first iteration beyond which $\kappa_{\Delta c2}\|J_k^T c_k\| \le \kappa_{\Delta c1}$ for $k \ge k_c$, which is well-defined because of (3.61). Lemma 3.9, (3.61) and (3.13) then imply that, for $k \in \mathcal{C} \setminus \mathcal{S}$, $k \ge k_c$ sufficiently large,

$$\Delta_k^c \ge \min\left[\kappa_{\Delta c2}\kappa_{\rm J}\|c_k\|\right] = \kappa_{\Delta c2}\kappa_{\rm J}\|c_k\|.$$

Observe also that Δ_k^c is maintained above $\kappa_{\Delta cc}\|J_k^T c_k\| \ge \kappa_{\Delta cc}\kappa_{\rm J}\|c_k\|$ at successful f-iterations because of (2.37) and (3.13) for k large enough. The ratio of Δ_k^c to $\|c_k\|$ also does not decrease at successful c-iterations, because then $\Delta_{k+1}^c \ge \Delta_k^c$ and $\|c_{k+1}\| \le \|c_k\|$. Moreover, Δ_k^c and $\|c_k\|$ do not change at unsuccessful f-iterations or y-iterations. Hence we deduce that

$$\Delta_k^c \ge \gamma_1 \min\left[\kappa_{\Delta cc}, \kappa_{\Delta c2}\right]\kappa_{\rm J}\|c_k\| \tag{3.96}$$

for all $k \ge k_c$ sufficiently large.

If now restrict our attention to $k \in \mathcal{C} \cap \mathcal{S} \subseteq \mathcal{A}$ (the last inclusion being guaranteed by part (ii) of Lemma 3.5), we obtain by using (2.39), (2.4), and (3.3) that

$$\theta_k - \theta_{k+1} \ge \eta_1 \kappa_{\rm cn}\kappa_{\rm nC}\|J_k^T c_k\| \min\left[\frac{\|J_k^T c_k\|}{\kappa_{\rm H}^2}, \Delta_k^c\right]. \tag{3.97}$$

Combining (3.96), (3.97), (3.13) and (2.1), we then obtain that for $k \in \mathcal{C} \cap \mathcal{S}$ sufficiently large,

$$\theta_k - \theta_{k+1} \ge 2\eta_1 \kappa_{\rm cn}\kappa_{\rm nC} \min\left[\frac{1}{\kappa_{\rm H}^2}, \gamma_1\kappa_{\Delta cc}, \gamma_1\kappa_{\Delta c2}\right]\kappa_{\rm J}^2\theta_k.$$

Thus (3.94) holds for $k \in \mathcal{C} \cap \mathcal{S}$ sufficiently large, with

$$\kappa_\theta \stackrel{\rm def}{=} 1 - 2\eta_1 \kappa_{\rm cn}\kappa_{\rm nC} \min\left[\frac{1}{\kappa_{\rm H}^2}, \gamma_1\kappa_{\Delta cc}, \gamma_1\kappa_{\Delta c2}\right]\kappa_{\rm J}^2 \in (0,1),$$

this last inclusion following from the definition of the various constants and particularly $\kappa_{\rm nC} \le \frac{1}{2}$.

We now observe that $\theta_k^{\max}$ is decreased in (2.41) at every successful c-iteration, yielding that, for $k \in \mathcal{C} \cap \mathcal{S}$ large enough,

$$\begin{aligned}
\theta_{k+1}^{\max} &= \max\left[\kappa_{\rm tx1}\theta_k^{\max}, \theta(x_k) - (1-\kappa_{\rm tx2})(\theta(x_k) - \theta(x_k^+))\right] \\
&\leq \max\left[\kappa_{\rm tx1}\theta_k^{\max}, \theta(x_k) - (1-\kappa_{\rm tx2})(1-\kappa_\theta)\theta(x_k)\right] \\
&\leq \max[\kappa_{\rm tx1}, 1-(1-\kappa_\theta)(1-\kappa_{\rm tx2})]\theta_k^{\max} \\
&= \kappa_{\theta m}\theta_k^{\max},
\end{aligned}$$

where we have used (3.94) and Lemma 2.2 to deduce the last inequalities, and where we have defined $\kappa_{\theta m} \stackrel{\rm def}{=} \max[\kappa_{\rm tx1}, 1-(1-\kappa_\theta)(1-\kappa_{\rm tx2})] \in (0,1)$. This yields (3.95) and concludes the proof. □

The penultimate step in our convergence analysis is to show that the variation in the objective function along the subsequence of c-iterations is bounded.

Lemma 3.29 *Suppose that $|\mathcal{C} \cap \mathcal{S}| = +\infty$ and that no subsequence exists such that (3.12) holds. Then*

$$\sum_{k \in \mathcal{C}} |\, f(x_k) - f(x_{k+1}) \,| < +\infty \tag{3.98}$$

and

$$\sum_{k \in \mathcal{C} \cap \mathcal{S}} \|s_k\| < +\infty. \tag{3.99}$$

Proof. Consider $k \in \mathcal{C} \cap \mathcal{S}$ and remember that Lemma 3.4 ensures that (3.14) holds for such a k. Using this property, the triangle inequality, (2.5) and (2.1), we verify that

$$\|s_k\| \leq \|n_k\| + \|t_k\| \leq (1+\kappa_{\mathcal{CS}})\|n_k\| \leq (1+\kappa_{\mathcal{CS}})\kappa_{\rm n}\|c_k\| = (1+\kappa_{\mathcal{CS}})\kappa_{\rm n}\sqrt{2\theta_k}. \tag{3.100}$$

Choose now k_0 large enough to ensure that (3.95) holds in Lemma 3.28 for all $k \geq k_0$. Then (3.100) yields that

$$\begin{aligned}
\sum_{k \in \mathcal{C}\cap\mathcal{S}, k \geq k_0} \|s_k\| &\leq (1+\kappa_{\mathcal{CS}})\kappa_{\rm n}\sqrt{2} \sum_{k \in \mathcal{C}\cap\mathcal{S}, k \geq k_0} \sqrt{\theta_k} \\
&\leq (1+\kappa_{\mathcal{CS}})\kappa_{\rm n}\sqrt{2} \sum_{k \in \mathcal{C}\cap\mathcal{S}, k \geq k_0} \sqrt{\theta_k^{\max}} \\
&\leq (1+\kappa_{\mathcal{CS}})\kappa_{\rm n}\frac{\sqrt{2\theta_{k_0}^{\max}}}{1-\sqrt{\kappa_{\theta m}}}
\end{aligned} \tag{3.101}$$

where we used (2.46) to deduce the second inequality, and the convergence of the geometric series implied by (3.95) and the monotonicity of the sequence $\{\theta_k^{\max}\}$ to deduce the third. This yields (3.99). As a consequence, we obtain that there is a $k_1 \geq k_0$ such that $\|s_k\| \leq 2$ for all $k \geq k_1$. Using now the mean-value theorem, we deduce that, for $k \in \mathcal{C} \cap \mathcal{S}$, $k \geq k_1$,

$$\begin{aligned}
|\, f(x_k) - f(x_{k+1}) \,| &= |\, \langle g_k, s_k\rangle + \tfrac{1}{2}\langle s_k, \nabla_{xx} f(\xi_k) s_k\rangle \,| \\
&\leq \kappa_{\rm H}\|s_k\| + \tfrac{1}{2}\kappa_{\rm H}\|s_k\|^2 \\
&\leq 2\kappa_{\rm H}\|s_k\|
\end{aligned} \tag{3.102}$$

for some $\xi_j \in [x_j, x_{j+1})$, and where we have used the Cauchy-Schwarz inequality, (3.1), and the bound $\|s_k\| \leq 2$. The bound (3.102), the inequality $k_1 \geq k_0$ and (3.101) then together yield that

$$\sum_{k \in \mathcal{C}\cap\mathcal{S}, k \geq k_1} |\, f(x_k) - f(x_{k+1}) \,| \leq 2\kappa_{\rm H} \sum_{k \in \mathcal{C}\cap\mathcal{S}, k \geq k_1} \|s_k\| \leq \frac{2(1+\kappa_{\mathcal{CS}})\kappa_{\rm n}\kappa_{\rm H}\|c_{k_0}\|}{1-\sqrt{\kappa_{\theta m}}}. \tag{3.103}$$

Note that this bound remains valid if $|\mathcal{C} \cap \mathcal{S}| < +\infty$ since the sum on the left-hand side is empty in that case. The desired conclusion immediately follows from the fact that $x_k = x_{k+1}$ for $k \in \mathcal{C} \setminus \mathcal{S}$. □

We finally strengthen the convergence results obtained in Theorem 3.22 by avoiding taking limits along subsequences.

Theorem 3.30 *Suppose that ω_t is strictly increasing in $[0, t_\omega]$ for some $t_\omega > 0$. Then, we have that, either there exists a subsequence indexed by $\mathcal{Z}$ such that (3.12) holds, or*

$$\lim_{k\to\infty} \|c_k\| = 0 \quad \textit{and} \quad \lim_{k\to\infty} \|P_k g_k\| = 0, \tag{3.104}$$

and all limit points of the sequence $\{x_k\}$ (if any) are first-order critical.

Proof. Assume that no subsequence exists such that (3.12) holds. If there are only finitely many successful iterations, the desired conclusion directly follows from Lemma 3.12. Assume therefore that $|\mathcal{S}| = +\infty$. If $|\mathcal{C} \cap \mathcal{S}| = \infty$, then the first limit in (3.104) follows from Theorem 3.22. On the other hand, if $|\mathcal{C} \cap \mathcal{S}| < \infty$, then the first limit in (3.104) follows from Lemma 3.27. Thus we only need to prove the second limit in (3.104) when there are infinitely many successful iterations.

To derive a contradiction, we assume that there exists an infinite subsequence indexed by $\mathcal{K}$ such that for some $\epsilon \in (0, \frac{1}{5})$

$$\min\left[\tfrac{1}{2}\|P_k g_k\|, \tfrac{1}{12}\|P_k g_k\|^2\right] \geq 10\epsilon \qquad \text{for all} \quad k \in \mathcal{K}. \tag{3.105}$$

Now choose $k_1 \in \mathcal{K}$ large enough to ensure that, for all $k \geq k_1$, (3.85) holds,

$$\|c_k\| \leq \kappa_{\rm c}, \tag{3.106}$$

and

$$\omega_t(\|c_k\|) \leq \tfrac{1}{2}\epsilon. \tag{3.107}$$

If $|\mathcal{C} \cap \mathcal{S}| = +\infty$, we also require that the conclusions of Lemma 3.28 apply for all k sufficiently large, and so that

$$\sum_{j=k_1, j\in\mathcal{C}\cap\mathcal{S}}^{\infty} \|s_j\| \leq \frac{\epsilon}{\kappa_{\rm H}^2(\kappa_{\rm P}+1)} \tag{3.108}$$

and

$$\sum_{j=k_1, j\in\mathcal{C}\cap\mathcal{S}}^{\infty} |\, f(x_j) - f(x_{j+1})\,| \leq \frac{\eta_1 \kappa_\delta \kappa_{\rm tC} \epsilon^2}{2\kappa_{\rm H}^2(\kappa_{\rm P}+1)}, \tag{3.109}$$

which is possible because of Lemma 3.29. Conversely, if $|\mathcal{C} \cap \mathcal{S}| < +\infty$, we require that k_1 is larger than the index of the last successful c-iteration (in which case (3.108) and (3.109) also hold since the sums on the left-hand sides are empty). Observe that, because of (3.85) and Lemma 3.25 (with $\alpha = 2$), (3.105) implies that

$$\pi_{k_1} \geq 2\epsilon > 0. \tag{3.110}$$

We now choose k_2 to be the (first) successful iteration after k_1 such that

$$\pi_{k_2} < \epsilon, \tag{3.111}$$

which we know must exist because of Theorem 3.22. Note that this last inequality, (3.85) and Lemma 3.25 (with $\alpha = 1$) then give that

$$\min\left[\tfrac{1}{2}\|P_{k_2} g_{k_2}\|, \tfrac{1}{12}\|P_{k_2} g_{k_2}\|^2\right] \leq 5\epsilon. \tag{3.112}$$

Our choice of k_1 and k_2 also yields that

$$\pi_j \geq \epsilon \quad \text{for} \quad k_1 \leq j < k_2. \tag{3.113}$$

We now observe that the objective function is decreased at every successful f-iteration and the total decrease, from iteration k_1 on, cannot exceed the maximum value of $f(x_k)$ for $k \geq k_1$ minus the lower bound f_{low} specified by (3.2). Moreover the maximum of $f(x_k)$ for $k \geq k_1$ cannot itself exceed $f(x_{k_1})$ augmented by the total increase occurring at all c-iterations beyond k_1, which is given by (3.109). As a consequence, we may conclude that

$$\begin{aligned}
\sum_{j=k_1, j\in\mathcal{S}}^{\infty} |\, f(x_j) - f(x_{j+1})\,| &= \sum_{j=k_1, j\in\mathcal{F}\cap\mathcal{S}}^{\infty} \left[\, f(x_j) - f(x_{j+1})\,\right] + \sum_{j=k_1, j\in\mathcal{C}\cap\mathcal{S}}^{\infty} |\, f(x_j) - f(x_{j+1})\,| \\
&\leq \left[f(x_{k_1}) + \frac{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon^2}{2\kappa_{\text{H}}^2(\kappa_{\text{P}}+1)} - f_{\text{low}} \right] + \frac{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon^2}{2\kappa_{\text{H}}^2(\kappa_{\text{P}}+1)},
\end{aligned}$$

which in turn implies that

$$\sum_{j=0, j\in\mathcal{S}}^{\infty} |\, f(x_j) - f(x_{j+1})\,| < +\infty \quad \text{and} \quad \lim_{\ell\to\infty} \sum_{j=\ell, j\in\mathcal{S}}^{\infty} |\, f(x_j) - f(x_{j+1})\,| = 0.$$

Because of this last limit, we may therefore possibly increase $k_1 \in \mathcal{K}$ (and k_2 accordingly) to ensure that

$$\sum_{j=k_1, j\in\mathcal{S}}^{\infty} |\, f(x_j) - f(x_{j+1})\,| \leq \min\left[\frac{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon^2}{2\kappa_{\text{G}}}, \frac{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon^2}{2\kappa_{\text{H}}^2(\kappa_{\text{P}}+1)} \right] \tag{3.114}$$

in addition to (3.85), (3.106), (3.107), as well as the conclusions of Lemma 3.28, (3.108) and (3.109).

Consider now a range of consecutive successful f-iterations (i.e. a range containing at least one successful f-iteration and no successful c-iteration), indexed by $\{k_a, \ldots, k_b-1\}$ lying between k_1 and k_2. Observe that (3.114) gives that

$$f(x_{k_a}) - f(x_{k_b}) \leq \frac{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon^2}{2\kappa_{\text{G}}}.$$

Then, using Lemma 3.26 (which is applicable because of (3.113) and this last bound), we deduce that

$$\|x_{k_a} - x_{k_b}\| \leq \frac{1}{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon}\left[f(x_{k_a}) - f(x_{k_b})\right].$$

We now sum on all disjoint sequences $\{k_{a,\ell}, \ldots, k_{b,\ell}\}_{\ell=1}^p$ of this type between k_1 and $k_2 - 1$ (if any), and find that

$$\sum_{j=k_1, j\in\mathcal{F}\cap\mathcal{S}}^{k_2-1} \|x_j - x_{j+1}\| = \sum_{\ell=1}^{p} \|x_{k_{a,\ell}} - x_{k_{b,\ell}}\| \leq \frac{1}{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon} \sum_{\ell=1}^{p} \left[f(x_{k_{a,\ell}}) - f(x_{k_{b,\ell}})\right]. \tag{3.115}$$

We now decompose this last sum and obtain, using (3.109) and (3.114), that

$$\begin{aligned}
\sum_{\ell=1}^{p} \left[f(x_{k_{a,\ell}}) - f(x_{k_{b,\ell}})\right] &\leq \sum_{j=k_1, j\in\mathcal{F}\cap\mathcal{S}}^{\infty} \left[f(x_j) - f(x_{j+1})\right] \\
&= \sum_{j=k_1, j\in\mathcal{S}}^{\infty} \left[f(x_j) - f(x_{j+1})\right] - \sum_{j=k_1, j\in\mathcal{C}\cap\mathcal{S}}^{\infty} \left[f(x_j) - f(x_{j+1})\right] \\
&\leq \sum_{j=k_1, j\in\mathcal{S}}^{\infty} |f(x_j) - f(x_{j+1})| + \sum_{j=k_1, j\in\mathcal{C}\cap\mathcal{S}}^{\infty} \left|f(x_j) - f(x_{j+1})\right| \\
&\leq \frac{\eta_1\kappa_\delta\kappa_{\text{tC}}\epsilon^2}{\kappa_{\text{H}}^2(\kappa_{\text{P}}+1)}
\end{aligned}$$

Substituting this inequality in (3.115), we obtain that

$$\sum_{j=k_1, j\in\mathcal{F}\cap\mathcal{S}}^{k_2-1} \|x_j - x_{j+1}\| \leq \frac{\epsilon}{\kappa_{\rm H}^2(\kappa_{\rm P}+1)}$$

and thus, using the triangle inequality and (3.108), that

$$\|x_{k_1} - x_{k_2}\| \leq \sum_{j=k_1, j\in\mathcal{C}\cap\mathcal{S}}^{k_2-1} \|x_j - x_{j+1}\| + \sum_{j=k_1, j\in\mathcal{F}\cap\mathcal{S}}^{k_2-1} \|x_j - x_{j+1}\| \leq \frac{2\epsilon}{\kappa_{\rm H}^2(\kappa_{\rm P}+1)}. \quad (3.116)$$

We now return to considering the sizes of the projected gradients at iterations k_1 and k_2. We know from the triangle inequality that

$$\begin{aligned} \|P_{k_1}g_{k_1}\| - \|P_{k_2}g_{k_2}\| &\leq \|P_{k_1}g_{k_1} - P_{k_2}g_{k_2}\| \\ &\leq \|(P_{k_1} - P_{k_2})g_{k_1}\| + \|P_{k_2}(g_{k_1} - g_{k_2})\| \\ &\leq \|P_{k_1} - P_{k_2}\|\,\|g_{k_1}\| + \|P_{k_2}\|\,\|g_{k_1} - g_{k_2}\|. \end{aligned}$$

In view of (3.106), we may now apply Lemma 3.23 and, recalling that the norm of an orthogonal projection is bounded above by one, deduce that

$$\|P_{k_1}g_{k_1}\| - \|P_{k_2}g_{k_2}\| \leq \kappa_{\rm P}\kappa_{\rm H}\|x_{k_1} - x_{k_2}\| + \|g_{k_1} - g_{k_1}\|, \quad (3.117)$$

where we have used (3.1) to bound $\|g_{k_1}\|$. But the vector-valued mean-value theorem ensures that

$$\begin{aligned} \|g_{k_1} - g_{k_2}\| &\leq \left\| \int_0^1 \nabla_{xx} f(x_{k_1} + t(x_{k_2} - x_{k_1}))(x_{k_1} - x_{k_2})\, dt \right\| \\ &\leq \max_{t\in[0,1]} \|\nabla_{xx} f(x_{k_1} + t(x_{k_2} - x_{k_1}))\|\,\|x_{k_1} - x_{k_2}\| \\ &\leq \kappa_{\rm H}\|x_{k_1} - x_{k_2}\|, \end{aligned}$$

where we also used (3.1). Substituting this last inequality in (3.117) and using (3.116), we finally obtain that

$$\|P_{k_1}g_{k_1}\| - \|P_{k_2}g_{k_2}\| \leq \kappa_{\rm H}(\kappa_{\rm P}+1)\|x_{k_1} - x_{k_2}\| \leq \frac{2\epsilon}{\kappa_{\rm H}}. \quad (3.118)$$

Observe now that the inequality $\epsilon \leq \frac{1}{5}$ and (3.112) imply together that

$$\|P_{k_2}g_{k_2}\| \leq 10\epsilon \leq 2 \quad \text{or} \quad \|P_{k_2}g_{k_2}\|^2 \leq 60\epsilon \leq 12 < 16,$$

which in turn implies that

$$\|P_{k_2}g_{k_2}\| < 4 \quad (3.119)$$

and thus that

$$\min\left[\tfrac{1}{2}\|P_{k_2}g_{k_2}\|, \tfrac{1}{12}\|P_{k_2}g_{k_2}\|^2\right] = \tfrac{1}{12}\|P_{k_2}g_{k_2}\|^2. \quad (3.120)$$

Suppose now that

$$\|P_{k_1}g_{k_1}\| \leq 6, \quad (3.121)$$

in which case

$$\min\left[\tfrac{1}{2}\|P_{k_1}g_{k_1}\|, \tfrac{1}{12}\|P_{k_1}g_{k_1}\|^2\right] = \tfrac{1}{12}\|P_{k_1}g_{k_1}\|^2. \quad (3.122)$$

Then, successively using (3.105), (3.112), (3.122), (3.120), the bound of one on the norm of orthogonal projections, (3.1) and (3.118), we conclude that

$$
\begin{aligned}
5\epsilon &\leq \min\left[\tfrac{1}{2}\|P_{k_1}g_{k_1}\|, \tfrac{1}{12}\|P_{k_1}g_{k_1}\|^2\right] - \min\left[\tfrac{1}{2}\|P_{k_2}g_{k_2}\|, \tfrac{1}{12}\|P_{k_2}g_{k_2}\|^2\right] \\
&= \tfrac{1}{12}\left[\|P_{k_1}g_{k_1}\|^2 - \|P_{k_2}g_{k_2}\|^2\right] \\
&= \tfrac{1}{12}\left[\|P_{k_1}g_{k_1}\| + \|P_{k_2}g_{k_2}\|\right]\left[\|P_{k_1}g_{k_1}\| - \|P_{k_2}g_{k_2}\|\right] \\
&\leq \tfrac{1}{6}\kappa_{\rm H}\left[\|P_{k_1}g_{k_1}\| - \|P_{k_2}g_{k_2}\|\right] \\
&\leq \tfrac{1}{3}\epsilon
\end{aligned}
$$

which is impossible. Hence (3.121) must be false. Combining now this observation with (3.118) and (3.119), we obtain that

$$2 < \|P_{k_1}g_{k_1}\| - \|P_{k_2}g_{k_2}\| \leq \frac{2\epsilon}{\kappa_{\rm H}},$$

which is again impossible because $\kappa_{\rm H} \geq 1 > \epsilon$. Hence our assumption (3.105) is itself impossible and the second limit of (3.104) must hold. □

3.4 Comments

We end our theoretical developments at this point, but the theory and results presented so far suggest some comments.

1. Although different from filter methods and penalty-type methods, the proposed algorithm unsurprisingly shares some of the main broad concepts used by these techniques to ensure global convergence.

   One can view the pair $(\theta(x_k), f(x_k))$ as some kind of temporary filter entry: an f-iteration from x_k needs not improve on feasibility, but should then result in progress on the objective function minimization, while, by contrast, a c-iteration allows the objective function to increase, but produces a significant decrease in infeasibility. This is very similar to what happens in filter methods, except that the filter entry is then remembered in the filter. By contrast, the pair is not stored in the trust-funnel method, but memory is instead provided by the decreasing nature of the sequence $\{\theta_k^{\max}\}$. Note that a similar mechanism is also included is some filter methods (see for Fletcher and Leyffer, 2002 for instance). It is also interesting to note that we have proved that every limit point of the sequence of iterates must be first-order critical, a result which has not been established for filter algorithms.

   The trust-funnel method is also related to penalty approaches, in that the decreasing bound $\theta_k^{\max}$ may possibly be interpreted as the effect of an increasing penalty parameter in this context. In this interpretation, the need to explicitly manage the parameter in the course of a penalty-based algorithm (which can be viewed as an indirect control on acceptable infeasibility) is replaced here by a more direct version of this control.

2. Assumption (3.2) is not really crucial in the sense that one may apply c-iterations (by temporarily setting $f \equiv 0$ and keeping $\hat{y}_k = 0$) *a priori* (hence reducing infeasibility) to reduce the domain. If a global lower bound on the objective function value on the feasible domain is known, a comparison of the infeasibility and objective function value at the starting point may be useful to decide whether pure c-iterations should be applied first, or if the complete algorithm can be applied directly from the starting point.

3. When the Jacobian J_k is of full-rank, we can rewrite the test (2.14) in the form

$$\|J_k(g_k^{\rm N} + J_k^T y_k)\| \le \omega_y(\|c_k\|),$$

which provides an implementable version of (2.14).

4. Convergence of trust-region methods for unconstrained optimization may be obtained as a by-product of the results presented here. Indeed, if there are no constraints, the algorithm reduces to the basic trust-region method by setting $\theta_0^{\max} = \kappa_{\rm ca}$, and, for every k, $n_k = 0$, $y_k = 0$, $\hat{y}_k = 0$, $r_k = g_k$. Since $\pi_k = \|g_k\|$, we have that $\pi_k > \omega_t(0) = 0$ and a non-zero t_k is always computed. Moreover, every iteration is then an f-iteration with $\delta_k^f = \delta_k^{f,t}$ at which we choose, as allowed by (2.37), not to update the (irrelevant) Δ_k^c.

5. Obviously, one could use $G_k = H_k$ and still obtain global convergence. The vector $\hat{y}_k$ then becomes irrelevant. This is particularly apt when the constraints are linear.

6. The tangential step is only required to satisfy the modified Cauchy condition (2.22), but there is no theoretical need to compute the associated modified Cauchy point (the solution of (2.19)). If one considers that t_k results from an iterative process starting (and possibly ending) at this modified Cauchy point, it is then necessary to ensure that this point satisfies either (2.26) or (2.23)-(2.24)-(2.25). A possible technique is to first solve (2.18) accurately enough to ensure that

$$\|c_k + J_k(n_k - \tau_k r_k)\|^2 \le \kappa_{\rm tt}\theta_k^{\max}, \tag{3.123}$$

which is possible since it holds trivially if (2.18) is solved exactly, because then $J_k r_k = 0$ by construction and $\vartheta_k < \|c_k + J_k n_k\|^2$. As soon as (3.123) holds, then the modified Cauchy point can be computed and (2.23) and (2.24) tested. If any of these fail, then the solution of (2.18) must be continued to ensure that

$$\|c_k + J_k(n_k - \tau_k r_k)\|^2 \le \vartheta_k$$

and a new, improved, modified Cauchy point can then be found along $-r_k$ at which (2.26) holds.

7. It is interesting to observe that the conditions (2.25) or (2.26) happen to be irrelevant for successful f-iterations in the theory discussed above. For such iterations, the role of limiting the acceptable infeasibility is played by (2.33).

   In a situation where evaluating the value of the infeasibility measure θ is cheap and the tangential step is computed by an iterative process, it may be possible to detect that (2.31) holds before the end of this process, and then simply replace conditions (2.25)/(2.26) by the verification that (2.33) holds. Of course, if (2.35) then fails or if (2.33) cannot be enforced, then the iteration has to be handled as an unsuccessful c-iteration, since we can no longer turn it into a successful c-iteration for which (2.25)/(2.26) is meaningful.

8. Preliminary numerical experience has shown that our algorithm, like many SQP methods, might suffer from the Maratos effect. A well documented cure for this problem (see Mayne and Polak, 1982, Coleman and Conn, 1982, or Section 15.3.2 of Conn et al., 2000) is to use second-order correction steps. In our context, we define such a step $s_k^{\rm C}$ as a step performed from $x_k + s_k$ to correct for an unsuccessful f-iteration, and such that

$$\|s_k + s_k^{\rm C}\| \le \Delta_k \tag{3.124}$$

   and

$$\theta(x_k + s_k + s_k^{\rm C}) \le \theta_k^{\max}. \tag{3.125}$$

Of course, for the f-iteration using the augmented step $s_k + s_k^{\rm C}$ to be successful , we still require, extending (2.35), that

$$\rho_k^{\rm C} \stackrel{\rm def}{=} \frac{f(x_k) - f(x_k + s_k + s_k^{\rm C})}{m_k(x_k) - m_k(x_k + s_k)} \geq \eta_1. \tag{3.126}$$

Using the comment just made on the irrelevant nature of (2.25) or (2.26) for successful f-iterations, we may now verify that the convergence theory presented above is not modified by the presence of these correction steps. Indeed, a successful iteration using the augmented step satisfies all the conditions required for a successful f-iteration where $m_k(x + s_k)$ is then interpreted, in the spirit of Section 10.4.2 in Conn et al. (2000), as a prediction of $f(x_k + s_k + s_k^{\rm C})$ and where the infeasibility-limiting condition (2.33) is replaced by (3.125).

In practice, a second-order correction is often computed by producing a step $s_k^{\rm C}$ that reduces infeasibility, typically by "projecting" the trial point lying in or close to the nullspace of $J(x_k)$ onto the actual feasible set. In this case, $s_k^{\rm C}$ not only improves feasibility (ensuring (3.125)), but often makes $m_k(x_k + s_k)$ to be a better prediction of the value of $f(x_k + s_k + s_k^{\rm C})$ than of $f(x_k + s_k)$ (which tends to make the iteration acceptable in (3.126)). Because $\|s_k^{\rm C}\|$ is then of the order of $\|s_k\|^2$, condition (3.124) usually follows from (2.47).

9. The first term in the maximum of (2.37) is only necessary to prove the "true limit" convergence result of Theorem 3.30, and, in this proof, only in the limit when $\|c_k\|$ converges to zero and $\|c_{k+1}\| > \|c_k\|$. Relaxed forms of (2.37) are therefore possible without affecting the theory developed above. It is also possible, in this update, to replace $\kappa_{\Delta{\rm cc}}$ by the line coordinate of the Cauchy step ν_{k+1}^u along the direction $-J_{k+1}^T c_{k+1}$, since this quantity is bounded below by $\kappa_{\rm H}^{-2}$. This strategy essentially amounts to choosing Δ_{k+1}^c large enough to allow the full Cauchy step at iteration $k+1$, which can be useful if $\|c_{k+1}\| > \|c_k\|$.

10. The authors anticipate that the convergence rate for the new method is essentially that which is known for composite step SQP methods (i.e., Q-superlinear or, under stronger assumptions, Q-quadratic). It seems most likely that either a second-order correction or a non-monotone acceptance rule will be required to obtain these results. The verification of this intuition is left for a future report.

The authors are well aware that many theoretical questions remain open at this stage of analysis, such as convergence to second-order critical points, rate of convergence, inequality constraints and worst-case complexity analysis. Furthermore, the many degrees of freedom in the algorithm provide considerable room for implementation tuning.

4 Conclusion and perspectives

We have presented a new SQP algorithm for the solution of the equality constrained nonlinear programming problem, that avoids the use of penalty parameters and that allows for inexact step computations. Convergence to first-order critical point has been proved.

A first line of work is the inclusion of a multi-dimensional filter mechanism (see Gould, Leyffer and Toint, 2005) in the algorithm, with the objective to make the constraint on decreasing infeasibility more flexible. Other non-monotone techniques, such as only requiring a decrease from the worst infeasibility over some past iterations could also be investigated. A second interesting development is the inclusion of bound or more general inequalities in the present framework. A third line of development is the design of a linesearch variant of the new method, possibly following ideas in Section 10 of Conn et al. (2000). On a more practical level, extensive numerical testing of the ideas presented here is necessary. These tests are ongoing, and preliminary results are encouraging.

Acknowledgments

N. Gould and Ph. Toint are grateful to Professor Y. Yuan and his students at the Chinese Academy of Sciences for the organization of the ICNAO2006 conference in Beijing, which provided an excellent environment for the derivation of some of the results presented here. They are also pleased to acknowledge the hospitality of CERFACS (Toulouse) where discussions with S. Gratton, D. Orban and A. Sartenaer stimulated the work that led to this paper. Many thanks are finally due to an anonymous thoughtful colleague, who provided input on the connections between the proposed method and existing techniques.

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