Poverty-Adjusted Life Expectancy: a consistent index in the quantity and the quality of life *

Jean-Marie Baland[†], Guilhem Cassan[‡], Benoit Decerf[§]

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Abstract

Poverty and mortality are arguably the two major sources of well-being losses. Most mainstream measures of human development capturing these two dimensions aggregate them in an ad-hoc and controversial way. In this paper, we develop a new index aggregating the poverty and the mortality observed in a given period in a consistent way. We call this index the poverty-adjusted life-expectancy ($PALE_\theta$). This indicator is based on a single normative parameter that transparently captures the trade-off between well-being losses from being poor or from being dead. We first show that $PALE_\theta$ follows naturally from the expected life-cycle utility approach a la Harsanyi (1953). Empirically, we then proceed to between countries or across time comparisons and focus on those situations in which poverty and mortality provide conflicting evaluations. Once we assume that being poor is (at least weakly) preferable to being dead, we show that about a third of these conflicting comparisons can be unambiguously ranked by $PALE_\theta$. Finally, we show that our index naturally defines a new and simple index of multidimensional poverty, the expected deprivation index.

JEL: D63, I32, O15.

Keyworks: Well-being index, Human development index, Multidimensional poverty, Poverty, Mortality.

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[†]CRED, DEFIPP, University of Namur.

[‡]CEPR, CRED, DEFIPP, University of Namur.

[§]World Bank, bdecerf@worldbank.org.

1 Introduction

Comparing well-being across societies in a simple, meaningful and unambiguous manner is a difficult task. The reason is that well-being is multidimensional. Looking at a dashboard of dimension-specific indicators is complex and typically yields a very partial ranking of societies. A summary index avoids these issues, but it is often meaningless and its comparisons are non-robust and therefore remain ambiguous. In this paper, we develop a new index of human well-being which focusses on two main dimensions, poverty and mortality. This index, called the poverty adjusted life expectancy, cumulates the following advantages which other competing indices typically fail to satisfy. First, it accounts for the multi-dimensional aspect of well-being in a straightforward manner by combining the two major ingredients of well-being, which are the quantity of life (through mortality) and its quality (through poverty). Being derived from a lifecycle utility approach, it is based on sound micro-foundations and is easily interpretable. Moreover, it takes into account distributional concerns, by focussing on individuals with low outcomes in each dimension. It also provides a ranking of societies in which mortality and poverty evolve in opposite directions. For a non trivial share of these comparisons, this ranking does not depend on the normative weight one gives to each dimension. Finally, our index does not require eliciting preferences and applies straighforwardly using available data.

There is a long-standing tradition looking for an indicator able to track the level of human development in a society (Hicks and Streeten, 1979; Stiglitz et al., 2009; Fleurbaey, 2009). Measuring well-being in a given period allows comparing human development across countries and across time. For this purpose, simple monetary measures, such as GDP per head, have been heavily criticized, essentially on two accounts.[1] First, income aggregates such as GDP are insensitive to the distribution of consumption across the population. This concern led to the design and adoption of income poverty measures (see, e.g., World Bank (2015)). Second, key aspects of human well-being, such as health, are extremely hard to meaningfully translate into monetary values. As a result, monetary measures do not provide a sufficient informational basis to account for the multi-dimensional nature of human development. They are therefore unfit to assess a country's performance at promoting well-being or to evaluate policies that imply trade-offs between different dimensions, e.g. environmental regulations or health policies.

Given these limitations, one strategy is to adopt instead a dashboard of indicators, such as the 17 Sustainable Development Goals (SDG) adopted in 2015 by the UN. One can also attempt to aggregate different dimensions of well-being into a single indicator of human development. Among these indicators, one finds the Human Development Index (HDI) (UNDP, 1990), the Level of Living Index (Drewnowski and Scott, 1966) or the Physical Quality of Life Index (Morris, 1978). Echoing the distributional concern, some of them focus on deprivations, like the Global Multidimensional Poverty Index (Alkire et al., 2015) or the Human Poverty Index (Watkins, 2006). These summary measures provide a rough yardstick of human development, which is arguably easier to communicate than a full list of various indicators. More importantly, they have the potential to solve the partial ranking of societies yielded

[1]Another important critique relates to sustainability of the well-being achieved in a particular period.

by a menu of indicators, when one society performs better along one dimension but not along another. A dashboard cannot indeed compare two societies when two or more dimensions are conflicting.

All these composite indices are subject to the same fundamental criticisms (Ravallion, 2011a,b; Ghislandi et al., 2019). First, the selection of the appropriate indicator in each dimension and the choice of the aggregation function are often arbitrary and do not follow from a defensible notion of individual well-being. This is particularly true for some dimensions, such as sanitation, which are essentially "inputs" into well-being, rather than "outcomes". Second, the system of weights embedded in the aggregation function is often arbitrary, for instance by giving an equal weight to each dimension. Such weights are typically not related to the choices individuals would make when facing a trade-off between these dimensions, and cannot therefore be taken as representative of human well-being. More fundamentally, different individuals may make different choices, which implies that no system of weights can be completely consensual or universal.

Taken together, these critics are devastating. Indeed, the full ranking of societies yielded by composite indices is of little value if the trade-offs they imply between "conflicting" dimensions is not meaningful. Moreover, the value of a summary indicator also depends on how quickly it can be grasped. Unfortunately, these indicators, originally conceived as pragmatic ordinal indicators, do not usually offer a simple interpretation that can be easily communicated.

Given these weaknesses, some scholars even argue in favor of reducing the informational basis to a unique dimension, such as health (Hicks and Streeten, 1979), thereby avoiding the need to choose a particular aggregation process. In this respect, a prominent and easily interpretable indicator is life-expectancy at birth, which can also be adapted in order to account for distributional concerns (Silber, 1983; Ghislandi et al., 2019; Gisbert, 2020). According to these authors, the cost of focussing on a single dimension may not be that high, as not all dimensions carry the same importance for human well-being.

In this paper, we propose to measure human well-being using the poverty-adjusted life-expectancy, $PALE_\theta$, a new summary index that aggregates well-being losses resulting from the poverty and mortality observed in a given period. There are good reasons to focus on poverty and mortality when measuring human development. First, poverty and mortality are arguably the two major sources of welfare losses: poverty entails welfare losses by reducing the quality of life while mortality entails welfare losses by reducing the quantity of life. Prominent scholars in welfare economics such as Deaton and Sen have dedicated a large part of their work to the study of poverty and mortality (Deaton, 2013; Sen, 1998). Unsurprisingly, the first two Sustainable Development Goals of the UN are directly related to poverty while the third one refers to mortality.[2] Second, focussing on poverty and mortality naturally reflects distributional concerns as they are the worst possible outcomes associated with consumption and health.

This summary index makes substantial progress on the criticisms identified above. In particular, the aggregation of poverty and mortality is normatively grounded on the expected life-cycle utility, the measure of social welfare proposed by Harsanyi

[2] The first two SDGs are entitled "No Poverty" and "Zero Hunger", while the majority of the indicators in the third "Good Health and Well-being" section refer to some form of mortality.

(1953). According to Harsanyi, social welfare in a given period can be understood as the life-cycle utility expected by a newborn when drawing at random a life that reflects the outcomes observed in that particular period. Our main simplification is to consider a binary quality of life: in any period, an individual is either poor or non-poor.[3] Life-cycle utility is then the sum of period utilities over one's lifetime, where period utility takes two values, one high when non-poor and one low when poor. Our index therefore normalizes the expected life-cycle utility when one expects, throughout her lifetime, to be confronted to the poverty and mortality prevailing in the current period. We call this index "poverty-adjusted life-expectancy" since, in a stationary society, this index simply counts the number of periods that a newborn expects to live but weighs down the periods that she expects to live in poverty. That is, our index has similar hypothesis and interpretation as the extremely popular life expectancy index. Mathematically, our index is obtained by multiplying life-expectancy at birth by a factor one minus the fraction of poor, where the fraction of poor is weighed down. This (normative) weight θ, the value of which lies between zero and one, corresponds to the fraction of the period utility lost when poor. When being poor has no utility cost, this weight takes the value zero and $PALE_0$ corresponds to life expectancy at birth. When being poor is as bad as losing one year of life, $\theta = 1$ and our index $PALE_1$ then corresponds to the poverty-free life-expectancy at birth (Riumallo-Herl et al., 2018), i.e. the number of years of life a newborn expects to live out of poverty.

As stressed above, some pairs of societies cannot be compared using a dashboard considering poverty and mortality separately because the two dimensions are "in conflict", for instance if one society has less poverty but higher mortality than the other. Even though our index relies in general on some weight given in the trade-off between poverty and mortality, we show that it can sometimes improve on this partial ranking for all plausible values of its weight, as long as one considers that being poor is not worse than being dead. (As we show below, a necessary and sufficient condition for an unambiguous ranking is that the index makes the same comparison for the two extreme values for its weight.) For instance, consider two societies A and B where B has a higher fraction of poor but a higher life-expectancy at birth. Suppose that the situation is such that one may expect to spend more periods in poverty in B than in A *but* also more periods out of poverty in B than in A, as people live longer in society B. It is easy to show that life-cycle utility is larger in B, regardless of the weight given to periods of poverty, because individuals on average live more periods of both types in B. Hence, provided that being poor is not worse than being dead, our index unambiguously ranks A and B, which a dashboard approach is unable to do. As a result, our index increases the set of pairs of societies that can be unambiguously compared. As long as the larger number of years spent in poverty is more than compensated by a longer life expectancy, $PALE_\theta$ and, therefore, welfare can only increase.

As we make clear later, our index, being closely related to the concept of life expectancy, is based on "expectations" whereby a newborn assumes to be exposed throughout his lifespan to the poverty and mortality observed in the current period.

[3] Clearly, we do not claim that our index is superior to Harsanyi's approach, but it is a plausible measure of expected life-cycle utility when considering poverty as the main factor reducing the quality of life. Also, the poverty status we consider here could also be a measure resulting from some aggregation of different dimensions of the quality of life.

It is therefore not a projection or a forecast of the average life-cycle utility of the cohort born in a particular period, implying that it cannot in general be interpreted as the expected life-cycle utility of a newborn, unless the society is stationary. However, even when mortality is selective and affects predominantly poor people, we also show that our index still provides a meaningful way to aggregate the two sources of welfare losses observed in a particular period. The reason why a risk-neutral social welfare function a la Harsanyi does not require to account for selective mortality is that mortality is a peculiar dimension as, once dead, all the other dimensions of deprivation become irrelevant. As a result, the aggregation of mortality and poverty is much simpler than the aggregation of other dimensions of deprivations affecting alive individuals.[4]

Empirically, we combine datasets provided by the World Bank data on income poverty (PovCalNet) and internationally comparable dataset on mortality data (the Global Burden of Disease) from 1990 to 2019. Again assuming that one year spent in poverty is (weakly) preferred to one year of life lost, we show that $PALE_\theta$ is able to solve a non-trivial number of ambiguous comparisons across time or between countries for which the two dimensions are conflicting. For instance, when comparing all possible pair of countries in each year, across all years, there are about 21 percent of such comparisons for which mortality and poverty move in opposite directions. Out of these ambiguous cases, $PALE_\theta$ is able to solve 35 percent of them. We also investigate the evolution of each country in the dataset, by comparing the situation in a particular year to that prevailing five years earlier. We find that, out of 27 percent of conflicting comparisons, $PALE_\theta$ is able to solve 38 percent of them.

Finally, we propose a generalization of our index that explicitly addresses distributional concerns about unequal lifespans. We define a new indicator of multidimensional poverty that also captures deprivation in the quantity of life, which requires the introduction of a normative age threshold below which one is considered as deprived, i.e. a definition of premature mortality. This new index, which we call the expected deprivation index ($ED_{\theta\hat{a}}$), is a weighted sum of the number of years that a newborn expect to lose prematurely and the number of years she expects to spend in poverty, using the same weight as in $PALE_\theta$. (Again, these expectations imply that a newborn assumes to be exposed throughout her lifespan to the poverty and mortality observed in the current period.) We show that this index enjoys the same advantages as $PALE_\theta$ and can usefully complement $PALE_\theta$ if one is concerned with unequal lifespans. In particular, it also increases the set of pairs that can be unambiguously compared when considering each dimension separately. In its spirit, $ED_{\theta\hat{a}}$ is similar to the Generated Deprivation index recently proposed by Baland et al. (2021), and they are in fact equal in stationary societies. We show that $ED_{\theta\hat{a}}$ is more reactive to contemporaneous policies (e.g. in the case of permanent mortality shocks), simpler to interpret and less data demanding than Generated Deprivation.

The poverty-adjusted life-expectancy is reminiscent of several indicators proposed in health economics, like the quality-adjusted life-expectancy (QALE) or the quality-adjusted life year (QALY).[5] Both account for the quality and quantity of life, by

[4]The mutual exclusivity of mortality and poverty simplifies their aggregation (Baland et al., 2021). In this paper, we show that a risk-neutral social welfare function justifies to first aggregate within each dimension and then aggregate across dimensions. In this sense, our index satisfies a form of path independence (Foster and Shneyerov, 2000).

[5]See for instance Whitehead and Ali (2010) for an economic interpretation of QALYs, or Heijink

weighting down the quantity of life for periods with low quality. They have been developed following the method of Sullivan (1971) and we show that these approaches directly follow from the expected life-cycle utility approach in stationary societies. Our index however accounts for another important dimension of well-being than health, which is poverty. Also, $PALE_\theta$ takes advantage of the existence of the well-established concept of a poverty threshold, which splits the population into poor and non-poor, thereby transforming the quality of life into a binary variable. This transformation is key to the simple interpretation of our index. There is, to the best of our knowledge, no immediate equivalent of such threshold in health economics.

There exist other indicators of a society's well-being which are arguably much superior to the one we propose. Yet, these indicators either rely on techniques that are not mature yet, require many arbitrary assumptions or cannot be readily applied on a large scale using existing data. For instance, Becker et al. (2005) and Jones and Klenow (2016) follow more sophisticated versions of Harsanyi's expected life-cycle utility approach by imposing a specific structure on preferences. Alternatively, Fleurbaey and Tadenuma (2014), in the case of well-being, or Decancq et al. (2019), for poverty, propose to aggregate different dimensions using individual preferences.[6] Also, there is a large litterature investigating the weights to be given to different dimensions of well-being (Benjamin et al., 2014; Decancq and Lugo, 2013). However, this literature has not reached full maturity, or cannot be applied on a large scale due to data constraints.

The remainder of the paper is organized as follows. In Section 2, we present the theory supporting our $PALE_\theta$ index and provide some empirical implications. In Section 3, we present the $ED_{\theta\hat{a}}$ index, which we compare to $PALE_\theta$ and Generated Deprivation. Section 4 concludes.

2 A transparent index of welfare

Our objective is to propose a simple indicator to measure and compare the level of human development of different societies in a given period. In particular, we would like this indicator to aggregate two major sources of welfare losses: mortality, which reduces the quantity of life, and poverty, which reduces the quality of life. This aggregation should follow from the way individuals aggregate these losses and therefore be related to life-cycle preferences.

The rationality requirements of decision theory provide a structure on admissible life-cycle preferences. Rational preferences over streams of consumption have been axiomatized by Koopmans (1960) and later generalized by Bleichrodt et al. (2008). Such preferences must be represented by a discounted utility function, which aggregates these streams as a discounted sum of period utilities

$$U = \sum_{a=0}^{d} \beta^a u(c_a) \tag{1}$$

where $d \in \mathbb{N}$ is the age at death, $\beta \in [0, 1]$ is the discount factor, c_a is consumption at age a and u is the period utility function.

et al. (2011); Jia et al. (2011) for applications of the QALE index to comparisons of health outcomes across populations.

[6]The limits of these different approaches are reviewed in Fleurbaey (2009).

Building on this representation of preferences, Harsanyi (1953) proposes to measure the welfare of a society by aggregating life-cycle utilities over the whole society. According to Harsanyi (1953), behind the veil of ignorance, each newborn faces a lottery whereby she ignores whether and when she will be poor and for how long she will live. When evaluating her life-cycle utility, she considers the life of a randomly drawn individual in that society. Following the formulation of Jones and Klenow (2016), her expected life-cycle utility is given by

$$EU = \mathbb{E} \sum_{a=0}^{a^*-1} \beta^a u(c_a) V(a), \tag{2}$$

where $V(a)$ is the (unconditional) probability that the newborn survives to age a, a^* is the maximal lifespan one can reach and the expectation operator $\mathbb{E}$ applies to the uncertainty with respect to consumption c_a. The period utility when being dead is normalized to zero. As a result, mortality is valued through its opportunity cost: death reduces the number of periods during which a newborn expects to consume.

Although this approach has solid theoretical foundations, it does not seem that the indicator defined by Eq. (2) could be directly used as a summary measure of human development. Indeed, this indicator requires the choice of a particular expression for the period utility function $u()$. Moreover, the trade-off between the quantity and quality of life that underlies this indicator depends on the definition of u and remains relatively obscure. And, finally, this indicator, being expressed in utility-units, does not lend itself to a direct interpretation.

2.1 The $PALE_\theta$ index

In order to improve on these issues, we consider two assumptions that simplify Eq. (2) into a simple index of human development. Our first simplifying assumption is to ignore discounting, i.e. $\beta = 1$. We argue that such assumption is necessary in order to assign equal weights to all individuals, regardless of their age. Indeed, Eq. (2) equates a society's welfare in a given period to the expected life-cycle utility of individuals born in that period. Clearly, the expected life-cycle utility of newborns is related to the society's welfare in a given period only when one assumes that their expected lives reflect at each age the outcomes observed for individuals of that age during the period considered. Discounting with a factor less than one would give less weight to the outcomes of older individuals.

Our second simplifying assumption is to transform consumption into a binary variable, i.e., c_a can be either being non-poor (NP) or being poor (P). This is obviously a strong assumption because we ignore the impact on period utility of consumption differences within these two categories. However, we argue that this assumption reflects the distributional concern, i.e. the desire to evaluate a society's development by focussing on the fate of its least well-off individuals. We believe this assumption is the price to pay when one wishes to focus on poverty as the main source of welfare losses, rather than other more general determinants of the quality of life.[7]

[7]This assumption is also used by Decerf et al. (2021) in a study of the poverty and mortality effects of the Covid-19 pandemic. These authors compare the relative sizes of poverty and mortality shocks, whereas we derive here an indicator of well-being.

Taken jointly, these two assumptions require the use of a simple indicator, which we call the poverty-adjusted life-expectancy ($PALE_\theta$). Our second assumption implies $\mathbb{E}u(c_a) = \pi(a)u_P + (1 - \pi(a))u_{NP}$ where $u_{NP} = u(NP)$, $u_P = u(P)$ and $\pi(a)$ is the probability to be poor at age a conditional on being alive at age a. As, by definition, life-expectancy at birth is $LE = \sum_{a=0}^{a^*-1} V(a)$, we can rewrite Eq. (2) as

$$EU = u_{NP}LE - (u_{NP} - u_P) \sum_{a=0}^{a^*-1} V(a)\pi(a). \tag{3}$$

In Section 2.3, we show that these two assumptions are sufficient to define $PALE_\theta$. We now provide a simple illustration showing how these two assumptions naturally lead to our index under a third assumption of "independence". Under the latter assumption, the conditional probability of being poor at each age a is a constant equal to the fraction of poor in the population, i.e., $\pi(a) = H$ for all $a \in \{0, \dots, a^* - 1\}$ where H is the head-count ratio. (Clearly, this independence assumption does not hold when mortality is selective, for instance when the poor die younger than the non-poor. We discuss this limitation in more details in Section 2.3.) We can then normalize Eq. (3) as

$$\frac{EU}{u_{NP}} = LE\Big(1 - \underbrace{\frac{u_{NP} - u_P}{u_{NP}}}_{\theta} H\Big).$$

This last expression defines the poverty-adjusted life-expectancy index:

$$PALE_\theta = LE(1 - \theta H). \tag{4}$$

The monotonicity of the period utility function implies that $u_{NP} \geq u_P$. Moreover, since being poor is not worse than being dead, we have $u_P \geq 0$. The parameter θ, which captures the fraction of utility lost when a non-poor individual becomes poor in a given period, is therefore such that $\theta \in [0, 1]$. Importantly, this parameter directly captures the trade-off between poverty and mortality. Indeed, as the period utility of being dead u_D is normalized to zero, we have $\frac{1}{\theta} = \frac{u_{NP}-u_D}{u_{NP}-u_P}$. Hence, the ratio $\frac{1}{\theta}$ measures, for a non-poor individual, the number of periods in poverty that are equivalent to being dead for one period.

$PALE_\theta$ has a simple expression: its first factor measures life-expectancy, whereas its second factor captures the fall in the quality of life due to poverty. This reduction depends on the value assigned to the parameter θ. When $\theta = 0$, becoming poor does not affect the quality of life and $PALE_0$ corresponds to life-expectancy at birth. When $\theta = 1$, being poor is equivalent to being dead and $PALE_1$ corresponds to the Poverty Free Life Expectancy (PFLE), an indicator proposed by Riumallo-Herl et al. (2018),which measures the number of years that an individual expects to live free from poverty.[8] For other values for θ, $PALE_\theta$ corresponds to the number of years of life free from poverty that provides the same life-cycle utility as that expected by a newborn.

$PALE_\theta$ aggregates a measure of mortality, LE, with a measure of poverty, H, in

[8] Riumallo-Herl et al. (2018) do not relate their PFLE index to a formal notion of social welfare. As our theory makes clear, the PFLE index reflects an extreme view on the trade-off between poverty and mortality, namely that being poor is as bad as being dead. One key difference between our work and Riumallo-Herl et al. (2018) is that, through our formal framework, we provide a sound theoretical basis for the aggregation process, even when mortality is selective, solve a number of "conflicting" situations and derive a parallel index of multidimensional deprivation.

a way consistent with life-cycle preferences. This is a progress over most composite indices, but $PALE_\theta$ also relies on a normative parameter, θ, that weights these two dimensions. Thus, one may wonder whether aggregating the two component indices is very useful given that there is a priori no consensus on the value that this parameter should take. Indeed, the welfare comparison of two societies based on $PALE_\theta$ may depend on the particular value assigned to the parameter θ. We show that a non-trivial part of these comparisons does not depend on the parameter value *even for some pairs not related by domination.* In other words, there exist pairs of societies such that one is poorer but the other has a higer mortality rate, that can be ranked by $PALE_\theta$ unambiguously, in the sense that this comparison holds for all admissible values of θ. Hence, the structure of expected life-cycle utility allows to extend comparisons beyond those associated to domination independently of the particular value assigned to θ.

We illustrate this property in Figure 1. Without aggregation, domination alone allows comparing society A with the NW quadrant (where societies have more poverty and more mortality) and the SE quadrant (where societies have less poverty and less mortality). For any value of θ, we can draw the iso-PALE$_\theta$ curves passing through A. The iso-PALE$_0$ curve (associated to $\theta = 0$) is a vertical line since poverty has no welfare costs and life expectancy is the sole determinant of welfare. Note however that the iso-PALE$_1$ curve (associated to $\theta = 1$) is not a horizontal line. This defines two additional areas for which welfare can be unambiguously compared with that of society A. The iso-PALE$_\theta$ curves associated to intermediate values of $\theta \in [0, 1]$ are indeed all located in the area between the iso-PALE$_0$ curve and the iso-PALE$_1$ curve. The area in the NE quadrant below the iso-PALE$_1$ curve yields an unambiguously higher welfare than A, even though these societies have a higher poverty than A. The area in the SW quadrant above the iso-PALE$_1$ yields an unambiguously lower welfare than A, even though these societies have a lower poverty than A. The size of these new areas depends on the marginal rate of substitution of $PALE_1$ at A. For society A and $PALE_1$, this marginal rate of substitution is given by $\frac{LE(A)(1-H(A))}{(LE(A))^2}$. If $LE(A) = 70$ and $H(A) = 20$, this marginal rate of substitution is equal to 0.011, meaning that one additional year of life is exactly compensated by an increase in the head-count ratio H of 1.1% percentage points.

We now provide some intuition for these additional unambiguous comparisons. They follow from (i) the fact that expected life-cycle utility sums period utilities and (ii) the assumption that a period in poverty is not worse than a period lost (i.e. $u_P \geq u_D$). For simplicity let us compare the life-cycle utility of two individuals i^A and i^B, who respectively live in societies A and B depicted in Figure 1. Assume that the larger poverty and smaller mortality of society B is such that the life of i^B has more periods in poverty than that of i^A, and the life of i^B also has more periods out of poverty than that of i^A. As both types of period are positively valued (ii), the value selected for the weight does not matter anymore. Indeed, i^B has a larger expected life-cycle utility because her life has more periods in each consumption status than the life of i^A. In other words, the larger poverty rates in society B is more than compensated by a longer life expectancy, so that an individual in society B always expect to live more years out of poverty than in society A. Conversely, in the SW quadrant above the iso-PALE$_1$, societies exhibit lower poverty rates but the fall in life expectancy in these societies is so large compared to society A that, despite the

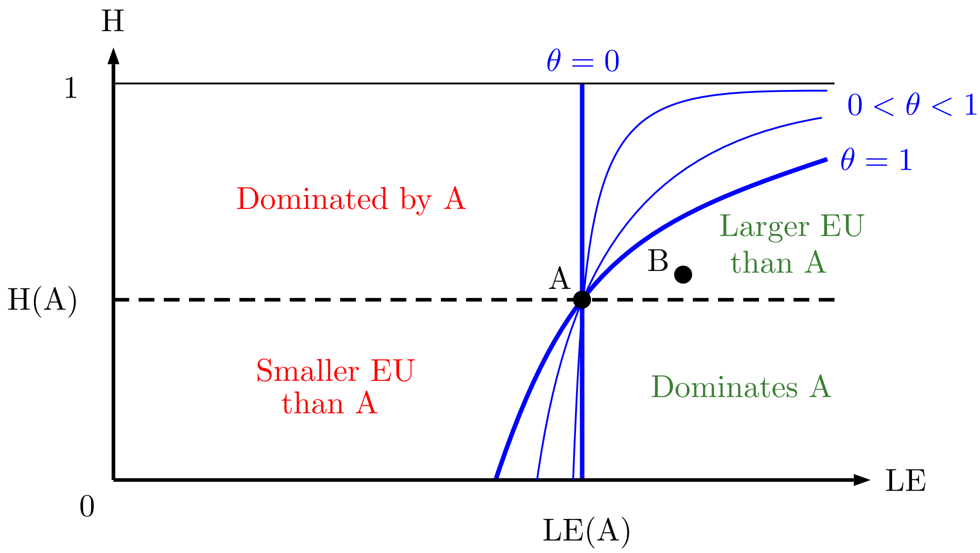


Figure 1: A simplified version of Harsanyi's expected life-cycle utility approach increases unambiguous comparisons.

redution in poverty rates, an individual expects to live less years out of poverty and society A is unambiguously preferred.

As an illustration, Table 1 below reports the situation of Pakistan and Bangladesh in 2019. Note that Life Expectancy can trivially be decomposed into Poverty Expectancy (LE*H) and PALE_1 (LE*(1-H)) which corresponds to Poverty Fee Life Expectancy. Pakistan has a lower headcount ratio than Bangladesh, but life expectancy is also lower in Pakistan. Therefore, it is a priori difficult to rank those two societies. Assuming that poverty and mortality remain unchanged, an individual born in Bangladesh can expect to spend 4.9 years of his life in poverty and 68.8 years out of poverty. In Pakistan, he can expect 2.8 years in poverty and 62.1 years out of poverty. Hence, a new born in Bangladesh can not only expect to spend more years poverty, but also more years out of poverty since the longer life expectancy there more than compensates for the higher poverty rate. As a result, $PALE_\theta$ ranks Bangladesh above Pakistan for all possible values of θ.

Table 1: An example of unambiguous comparison: Pakistan and Bangladesh in 2019.

	Headcount ratio	Life Expectancy	Poverty Expectancy $(LE * H)$	Poverty Free Life Expectancy $LE * (1 - H) = PALE_1$
Pakistan	4.3%	64.8	2.8	62.1
Bangladesh	6.7%	73.6	4.9	68.8

Ignoring mortality leads to correct welfare comparisons whenever there is domination, meaning that H and LE yield the same ranking. Theses cases correspond to the NW and SE quadrants in Figure 1. (Clearly, when H yield the same ranking as LE, $PALE_0$ automatically yields the same ranking as $PALE_1$.) In the absence of domination (NE and SW quadrants in Figure 1), ignoring mortality may lead to erroneous welfare comparisons. First there are cases such that ignoring mortality always lead to unambiguously wrong comparisons, independently of the value assigned to he normative parameter θ. In Figure 1, these cases correspond to the areas in the NE and SW quadrants that are between the iso-PALE$_1$ curve and the dashed horizontal line. For other cases, disregarding mortality leads to correct or incorrect comparisons depending on the value of θ. This occurs when $PALE_0$ and $PALE_1$ yield opposite rankings, which corresponds in Figure 1 to the areas between the iso-PALE$_0$ and iso-PALE$_1$ curves. Proposition 1 provides the conditions under which ignoring mortality, i.e., comparing two societies based on H, always leads to wrong welfare comparisons.

Proposition 1. *(Unambiguous comparisons of welfare)*
(i) For any two societies A and B, $PALE_\theta(A) < PALE_\theta(B)$ for all $\theta \in [0,1]$ if and only if

$$PALE_0(A) < PALE_0(B) \text{ and } PALE_1(A) < PALE_1(B) \qquad \text{(Condition C1)}$$

(ii) There exist societies A and B for which $PALE_\theta(A) < PALE_\theta(B)$ for all $\theta \in [0,1]$ even though $H(A) < H(B)$. These societies are such that $H(A) < H(B)$ and $LE(A) < LE(B)$.

Proof. See Appendix 5.1. □

2.2 Applications of $PALE_\theta$

The data on population and mortality by country, age group and year comes from the Global Burden of Disease database (2019). Comparable information across countries and over time is available for the 1990-2019 period and is, to our knowledge, the most comprehensive mortality data available for international comparison.[9] Data on alive deprivation come from the PovcalNet website which provides internationally comparable estimates of income deprivation level. This data set is based on income and consumption data from representative surveys carried out in low- and middle-income countries between 1981 and 2019.[10] In our empirical application, we follow the World Bank's definition of extreme income deprivation, corresponding to the \$1.9 a day threshold (Ferreira et al., 2016). We merged the two databases at the year and country level. Since the Global Burden of the Disease data are only available since 1990, we focus on the 1990-2019 period for a total of 120 low- and middle-income countries.

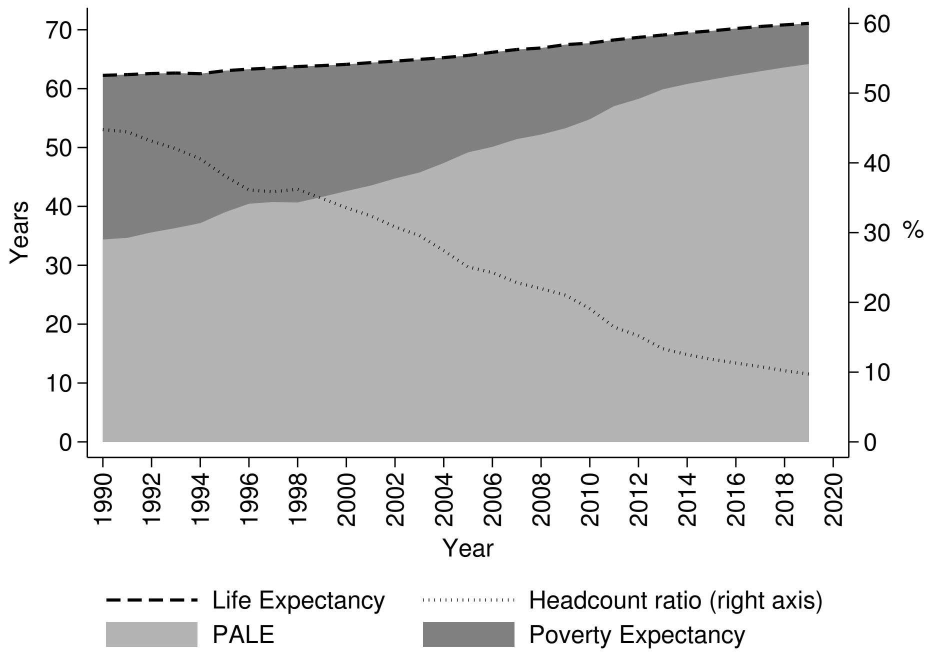
We first present in Figure 2 the evolution of life expectancy, the headcount ratio and $PALE_\theta$ for these countries during the period 1990-2019. When $\theta = 1$, life expectancy can be trivially decomposed into poverty expectancy and poverty adjusted life expectancy: the difference between LE and $PALE_1$ is the number of years a

[9]To construct this database, population and mortality data are systematically recorded across countries and time from various data sources (official vital statistics data, fertility history data as well as data sources compiling deaths from catastrophic events). These primary data are then converted into data in five years age groups, at year and country level using various interpolations and inference methods (see Global Burden of Disease Collaborative Network (2020) for more information on the GBD data construction). Following the literature, we only consider the point estimate in the number of deaths (see also Hoyland et al. (2012) for a critique of this approach).

[10]The website address is http://iresearch.worldbank.org/PovcalNet/povOnDemand.aspx. Each country's income deprivation level in PovCalNet is computed on a three year basis, and yearly data are obtained by linear interpolation. In order to keep the panel balanced, we also extrapolate the data and keep countries for which only 5 years or less of data have been extrapolated. A more detailed description of the data source is given in Chen and Ravallion (2013).

newborn expects to live in poverty. (For $\theta < 1$, the corresponding $PALE_\theta$ curves all lie between life expectancy and the $PALE_1$ curve.) Throughout the period, life expectancy increased from 62.3 in 1990 to 71.1 in 2019 but the decrease in poverty expectancy is even more spectacular, from 27.9 years in 1990 down to 6.9 years in 2019. This decrease in poverty combined with an increase in life expectancy resulted in a large increase in $PALE_1$, from 34.4 in 1990 to 64.2 years in 2019.

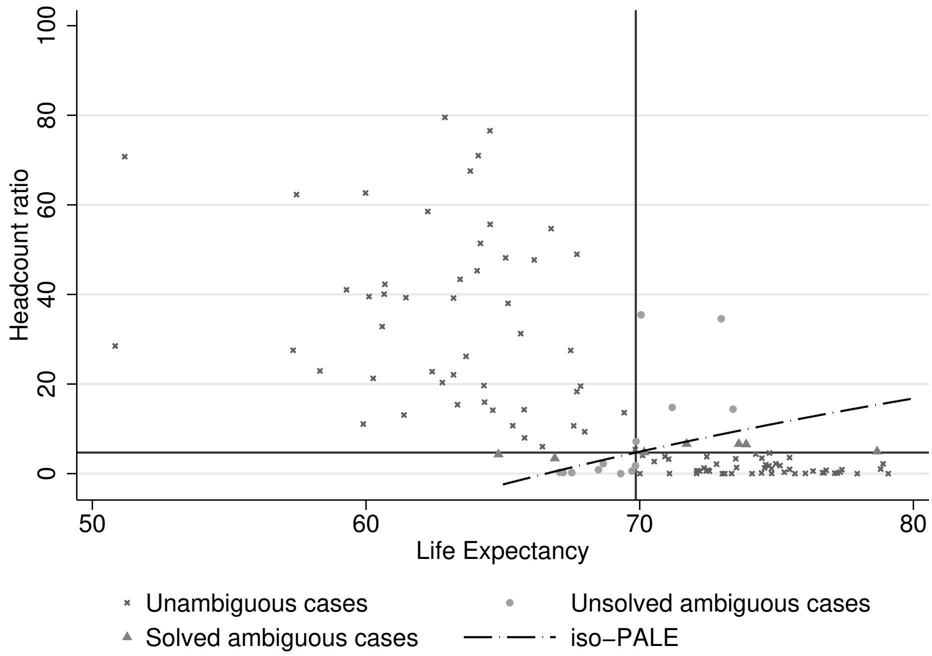
Figure 2: Evolution of $PALE_1$ and Life Expectancy, 1990-2019



We now attempt to quantify the value added of the $PALE_\theta$ index as compared to a menu of two separate indicators (LE and H). To do this, we quantify the number of situations for which the two indicators are "in conflict" and the percentage of these conflicting' situations that are unambiguously ranked by $PALE_\theta$. We again assume that the weight θ is equal to one, which corresponds to the most conservative approach consistent with the idea that being poor is weakly preferable to being dead. (Choosing a lower maximal value for θ, by decreasing the maximal weight given to the poverty component, would mechanically increase the number of situations that we can unambiguously compare with $PALE_\theta$.)

We first provide an empirical version of Figure 1 above by comparing the situations of different countries in 2019. The resulting diagram is presented in Figure 3 below. The point of reference (point A in Figure 1) chosen for this diagram is defined as a hypothetical reference country with a median head count ratio and a median life-expectancy at birth, which corresponds roughly to the situation of Nepal in 2019. The iso-$PALE_1$ curve is represented by the dotted curve. All countries below this iso-$PALE_1$ curve have a larger $PALE_1$ value than the reference country. Among these, some countries, located in the south east quadrant, are obviously better off, with a larger life-expectancy and lower poverty levels. Others, located in the north west quadrant, are unambiguously worse off. In the other two quadrants, there is a significant number of countries for which the evolution of life-expectancy and poverty are conflicting as they go in opposite directions. Among these, those represented by shaded triangles correspond to situations in which the comparison by $PALE_{\theta}$ is unambiguous. In the north-east quadrant, $PALE_{\theta}$ is always larger, as higher poverty is more than compensated for by lower mortality. In the south-west quadrant, $PALE_{\theta}$ is unambiguously smaller, as the fall in poverty is not large enough to compensate for the higher mortality. Countries represented by a small dots are countries we cannot rank unambiguously, as this ranking depends on the particular value assigned to θ.

Figure 3: Comparing countries in 2019



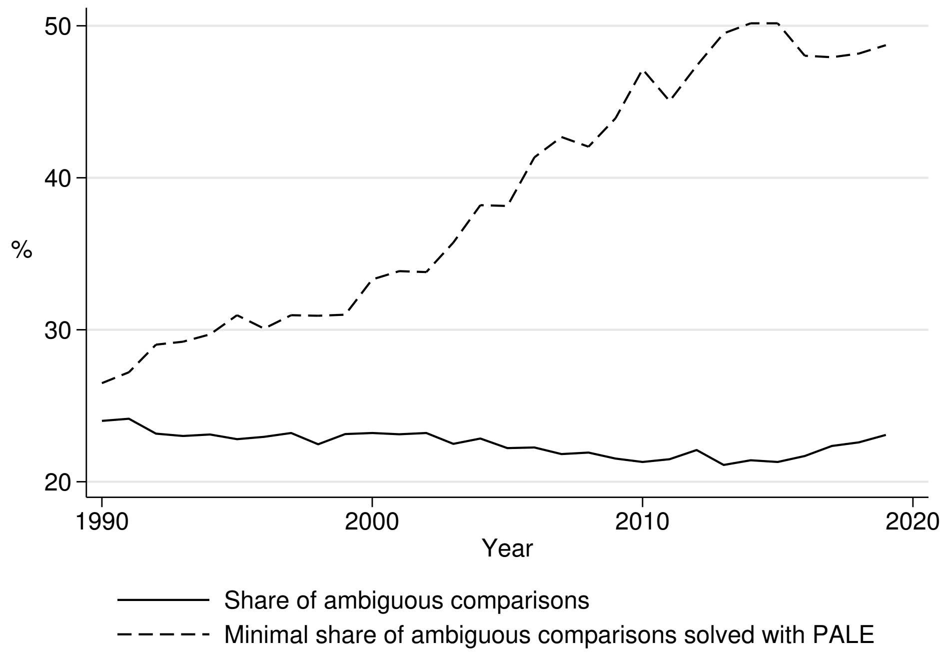
(a) 2019

Note: for the sake of presentation, we only report in the figure observations for which life-expectancy is larger than 40.

Figure 4 replicates this exercice by comparing all pairs of countries for each year between 1990 and 2019, and reports, among all these comparisons, the proportion of cases which are ambiguous, and the share of these ambiguous cases for which $PALE_{\theta}$ provides a unambiguous answer. Out of 23 percent of ambiguous comparisons, $PALE_{\theta}$ is able to solve at least 37 percent of them. The share of ambiguous comparisons that our index unambiguously solves strongly increases overtime due to the falling incidence of absolute poverty in many countries.

In Figure 5, we provide $PALE_{\theta}$ comparisons within countries between present and past situations. More precisely, for each year, we compare the situation in period t to the situation prevailing in the same country five years earlier. Given that each country's situation changed over time, we need to adapt our graphical presentation to represent the set of situations for which $PALE_{\theta}$ stays constant over time. We again conservatively assume θ equal to one.

Figure 4: Evolution of the resolution of ambiguous inter-country comparisons, 1990-2019



Reading: in 1990, countries had on average 23% of ambiguous comparisons, out of which at least 26% were solved by the use of PALE.

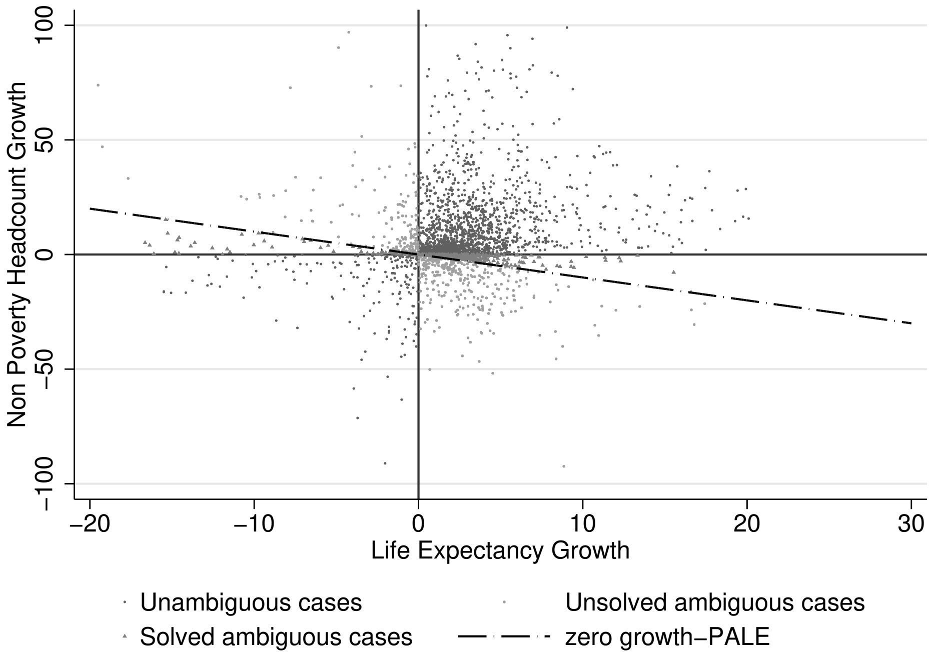
By definition, $PALE_1 = LE(1-H)$, and thus $PALE_1$ increases if and only if $dLE/LE > d(1-H)/(1-H)$. This simple expression allows us to contruct a figure in the $(dLE/LE, d(1-H)/(1-H))$ plan, in which the rate of growth of LE is measured on the horizontal axis, and the rate of growth of $(1-H)$, which we refer to as the "Non-poverty Headcount", on the vertical axis.[11] We define the "zero-growth $PALE_1$" curve, which represents all the combinations of the two growth rates such that $PALE_1$ remains unchanged: $dLE/LE = d(1-H)/(1-H)$. Above this curve, $PALE_1$ increases and below this curve $PALE_1$ decreases.

The situations of interest are located in the North West and in South East quadrants in which the two indicators move in opposite directions. In these quadrants, there are two regions, one in the triangle below the curve in the north west quadrant, and one in the triangle above the curve in the south east quadrant for which $PALE_\theta$ is able to provide a clear welfare comparison. In these two areas, the shaded triangles represent situations in which, in a particular country, the situation either strictly improved (in the south east quadrant) or deteriorated (in the north west quadrant) compared to the situation prevailing in the same country five years earlier.[12]

[11] For the sake of the graphical presentation, we excluded from the graph measures that could be considered as outliers (growth in non poverty headcount larger or smaller than 100 percent, and growth rates in life-expectancy larger than 90 percent or smaller than -40 percent). These are however adequately accounted for in the following graph.

[12] Again, if being dead is strictly worse than being poor, so that θ is always strictly lower than one, more situations can be strictly signed. They are located in the triangle above the "zero-growth $PALE_1$" in the NW quadrant, and in the triangle below the "zero-growth $PALE_1$" in the SE quadrant.

Figure 5: Resolution of ambiguous countries' evolutions, 1990-2019



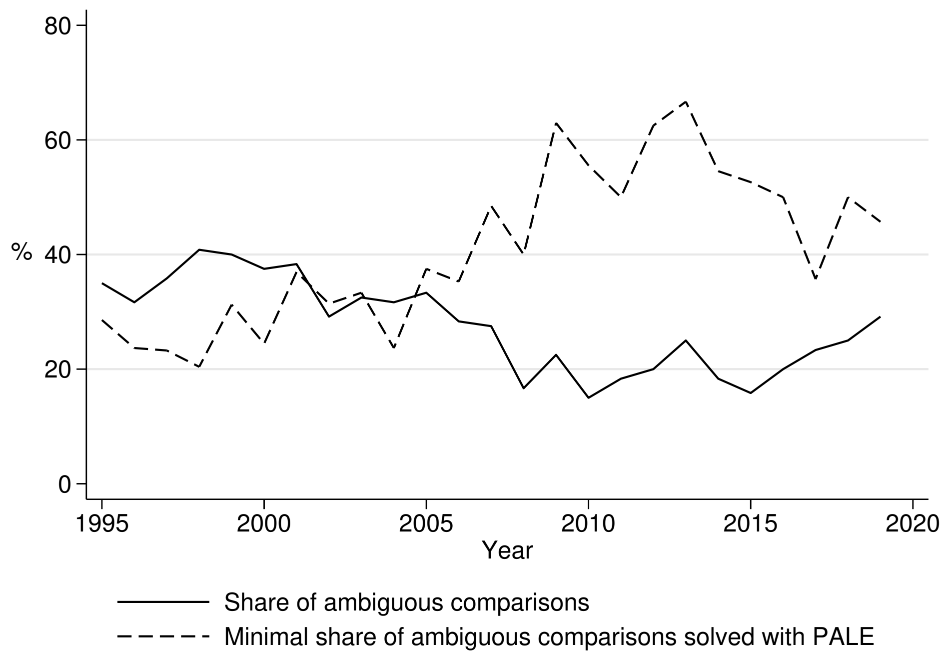
Finally, Figure 6 reports, using the same comparisons, the evolution over time of the share of ambiguous situations in which life-expectancy and poverty moved in opposite directions in one country between t and t+5, and the share of these ambiguous situations for which the most conservative definition of $PALE_\theta$ provides a clear ranking. Overall, the share of ambiguous comparisons declines from about 30 to 20 percent over the period considered (with an overall average of 27 percent). Out of these, we can solve an average of 38 percent of welfare comparisons, from about 20 in 1990 to more than 50 percent in the last years considered.

2.3 $PALE_\theta$ beyond the independence case

We have shown in Section 2.1 that $PALE_\theta$ corresponds to a simplified version of expected life-cycle utility (Eq. (3)) under the assumption of independence. However, independence is unlikely in practice: mortality is selective as the poor die younger than the non-poor (Chetty et al., 2016). Canudas-Romo (2018) points to this limitation when criticizing the PFLE index of Riumallo-Herl et al. (2018), which corresponds to $PALE_1$. This may cast some doubts on whether $PALE_\theta$ is a valid measure of the welfare losses suffered in a given period. In this section, we show that, in the absence of independence, $PALE_\theta$ still corresponds to the expected life-cycle utility in any stationary society. We then argue that this result provides the conceptual foundation for the use of $PALE_\theta$ even in societies that are not stationary.

The particularity of a stationary society is that all outcomes observed in one period are replicated in the following period. In our setting, a society is stationary if natality, mortality and poverty are constant over time. As a result, in a stationary society, one can perfectly infer his expected life-cycle utility from the mortality and

Figure 6: Evolution of the resolution of ambiguous countries' evolutions, 1990-2019



poverty prevailing at the time of his birth. $PALE_\theta$ is then a simple normalization of his expected life-cycle utility, even when the independence assumption does not hold.[13]

Proposition 2 (Correspondence between Harsanyi and $PALE_\theta$)**.**
For any stationary society, $PALE_\theta = \frac{EU}{u_{NP}}$.

Proof. A formal statement and proof is provided in Appendix 5.2. □

Clearly, in practice, populations are not stationary and we cannot in general interpret $PALE_\theta$ as the expected life-cycle utility of a newborn. Indeed, the poverty and mortality observed at birth might not be good predictors for the future, in particular as mortality and mortality decline over time with medical progress or economic growth. Therefore, $PALE_\theta$ should *not* in general be interpreted as a projection or a forecast of the average life-cycle utility that the cohort of individuals born in the period will enjoy during their lives. This being said, the validity of $PALE_\theta$ as an indicator of a society's welfare in period t does *not* rely on whether this indicator correctly forecasts the future. Our objective is to aggregate the welfare losses observed in period t using a lifecycle utility approach. This aggregation should not depend on the future evolutions of poverty and mortality.[14] Rather, the way to aggregate the welfare losses in period t that is consistent with a lifecycle utility

[13]We discuss in the conclusion how to adapt $PALE_\theta$ if the social planner is not indifferent to the fact that some individuals cumulate poverty and early mortality, which typically happens when mortality is selective. Harsanyi's social welfare as defined in Eq. (2) is indifferent to such cumulation.

[14]For instance, a transitory mortality or poverty shock – due to war or to another disaster – does reduce current welfare, even if the country fully recovers in the next period. In contrast, the transitory nature of the shock implies that its consequences affect essentially the current generations. Its impact on the actual expected life-cycle utility of newborns can therefore be negligible, or nil if the shock did not affect the mortality rates of the newborns.

approach is to take the perspective of a newborn who *assumes* that she is born in a stationary population, i.e. that the poverty and mortality observed at the time of her birth remain unchanged during her whole life. Proposition 2 shows that $PALE_\theta$ is a normalization of the expected life-cycle utility of a newborn who makes this assumption. In other words, even if we had a perfect forecast of the future average lifecycle utility of individuals born in a non-stationary society, $PALE_\theta$ provides a much better picture of the welfare losses in the period of their birth.

It is worth noting that the same point can be made about life-expectancy at birth (LE). In practice, this measure is derived from the mortality vector observed in a given period. As a result, this index does not correspond to the average lifespan of a cohort born in that period if the society is not stationary. However, life expectancy corresponds to the number of years of life that a newborn expects to live when she assumes she is born in a stationary society. The way it aggregates current mortality rates is widely accepted as a meaningful measure of period mortality.

3 A transparent index of deprivation

The normative relevance of one's death may depend on the age at which death occurs. This judgment is implicit in several mainstream multidimensional indicators. For instance, the global Multidimensional Poverty Indicator only accounts for deaths below 18 years old (Alkire et al., 2015), or the Human Poverty Index only accounts for deaths below 40 years old (Watkins, 2006). The widespread focus on child mortality follows the same logic.

This shows that one normative limitation of $PALE_\theta$ is that it does not reflect the distributional concern in both its dimensions. Although $PALE_\theta$ does focus on a low quality of life due to poverty, $PALE_\theta$ does not particularly focus on a low quantity of life. Indeed, an additional year of life given to an old individual has the same impact on $PALE_\theta$ as an additional year of life given to a young one. It is true that, in general, lifespans are distributed less unequally than consumption (Peltzman, 2009), which slightly tunes down the need to capture unequal lifespans when monitoring human development. Nevertheless, concerns around unequal lifespans justify the use of an indicator that is sensitive to very low lifespans.

In this section, we extend our welfare index to measure deprivation in both the quality and quantity of life. As stressed in the introduction, multidimensional poverty indices capturing the quality and quantity of life are plagued by the same limitations as welfare indices. They typically lack solid theorethical foundations and black box opaque trade-offs (Ravallion, 2011b).

Two properties of a measure of deprivation require us to adapt the $PALE_\theta$ index. First, we must define deprivation in the quantity of life. Borrowing from a long tradition focussing on absolute poverty, we consider as deprived an individual who dies prematurely, i.e. who dies before reaching a minimal age threshold $\hat{a}$. Following Baland et al. (2021), we call "lifespan deprived" an individual who dies before reaching this age threshold. Second, deprivation is the opposite concept of welfare, i.e. deprivation decreases when welfare increases. As we now show, under these two properties, $PALE_\theta$ naturally leads to a particular index of deprivation.

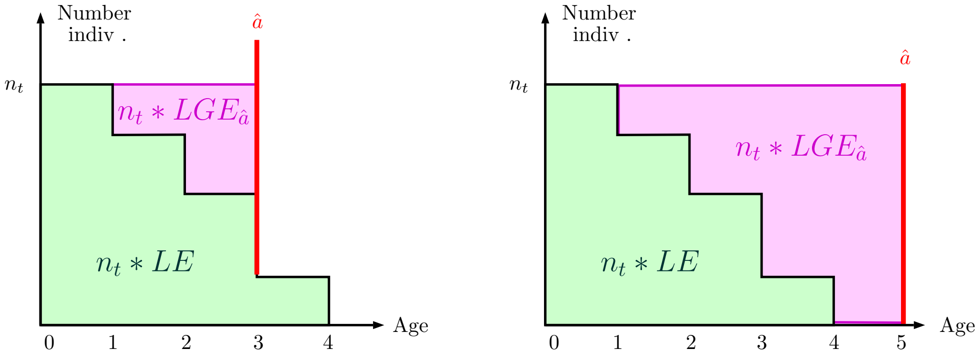
3.1 Expected deprivation: the $ED_{\theta\hat{a}}$ index

We call expected deprivation at birth ($ED_{\theta\hat{a}}$) the generalization of index $PALE_\theta$ in a deprivation context. The main difference is that $ED_{\theta\hat{a}}$ is based on an indicator of mortality different from LE. Indeed, when focusing on deprivation in the quantity of life, only the years of life lost *before* reaching the minimal age threshold $\hat{a}$ matter. We therefore define another indicator of mortality, the lifespan gap expectancy, which measures the number of years that a newborn expects to lose prematurely.[15] Letting $n_t(a)$ denote the number of individuals born in period t who survive at least to age a and $n_t = n_t(0)$, we have:[16]

$$LGE_{\hat{a}} = \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \frac{n_t(a) - n_t(a+1)}{n_t}.$$

We illustrate in Figure 7 the close connection between $LGE_{\hat{a}}$ and LE, where $LE = \sum_{a=0}^{a^*-1} \frac{n_t(a)}{n_t}$. In the figure, we construct a counterfactual population pyramid by reporting for each age a the number $n_t(a)$ of newborns who are still alive at age a. As explained in Section 2.3, this counterfactual pyramid corresponds to the population pyramid in period t if the society is stationary in period t.[17] In the left panel of Figure 7, LE is proportional to the area below the population pyramid. By contrast, $LGE_{\hat{a}}$ is proportional to the area between this pyramid and the age threshold. The right panel illustrates the property that, for large enough age thresholds, $LGE_{\hat{a}}$ is the complement of LE. Formally, when $\hat{a} \geq a^*$, $LGE_{\hat{a}} = \hat{a} - LE$.

Figure 7: Life Expectancy and Lifespan Gap Expectancy



Note: in the Left panel, the light grey area below the counterfactual "stationary" population pyramid is a multiple of LE and the dark grey area is a multiple of $LGE_{\hat{a}}$.

The expected deprivation index, $ED_{\theta\hat{a}}$, aggregates the poverty and lifespan deprivation expected by a newborn, if she considers facing, throughout her life-cycle, the poverty and mortality prevailing at the time of her birth. It combines a component for deprivation in the quality of life and a component for deprivation in the

[15] $LGE_{\hat{a}}$ is a particular version of the Years of Potential Life Lost, an indicator used in medical research in order to quantify and compare the burden on society due to different death causes (Gardner and Sanborn, 1990).

[16] See Proposition 5 for a mathematical expression for $LGE_{\hat{a}}$ that only depends on the mortality observed in period t. See also Appendix 5.2 for more details on the formal framework.

[17] In a stationary society, the current population pyramid can be obtained by successively applying the current age-specific mortality rates to each age group.

quantity of life:

$$ED_{\theta\hat{a}} = \underbrace{\frac{LGE_{\hat{a}}}{LE + LGE_{\hat{a}}}}_{quantity\ deprivation} + \theta \underbrace{\frac{LE * H}{LE + LGE_{\hat{a}}}}_{quality\ deprivation}, \quad (5)$$

where the parameter $\theta \in [0, 1]$ is defined in exactly the same way as for $PALE_{\theta}$ and the age threshold $\hat{a} \in \mathbb{N}_0$ must respect a lower-bound $\underline{\hat{a}} \in \mathbb{N}_0$, such that $\hat{a} \geq \underline{\hat{a}} \geq 0$. The value for the lower bound $\underline{\hat{a}}$ influences the set of comparisons that are robust to the values selected for θ and $\hat{a}$ (see below).

The two normative parameters θ and $\hat{a}$ jointly define the respective importance attributed to poverty and mortality. Parameter θ determines the relative weights of being dead or being poor for one period. In contrast, parameter $\hat{a}$ determines the number of periods for which "being prematurely dead" is accounted for by the index. Hence, $\hat{a}$ also affects the relative sizes of these two sources of welfare losses throught its impact on $LGE_{\hat{a}}$.

Both components have the same denominator, which measures a normative lifespan corresponding to the sum of LE and $LGE_{\hat{a}}$. This normative lifespan can be interpreted as the (counterfactual) life-expectancy at birth that would prevail if all premature deaths were postoned to the age threshold. It is at least as large as LE, and corresponds to LE if the age treshold is equal to 0. The numerator of each term measures the expected number of years characterized by one of the two dimensions of deprivation. The numerator of the quantity deprivation component measures the number of years that a newborn expects to lose prematurely (when observing mortality in the period) given the age threshold, $\hat{a}$. The numerator of the quality deprivation component measures the number of years that a newborn expects to spend in poverty.

As said above, these expectations are correct in the case of independence (see Section 2.1) or in stationary societies (see Section 2.3). As discussed above, in a stationary population, the poverty and mortality rates prevailling at its birth perfectly reflect the poverty and mortality a new cohort will be confronted to in the future. This restriction does not however invalidate the use of $ED_{\theta\hat{a}}$ as an indicator of deprivation in the current period (see Section 2.3): again, a widely used index such as life-expectancy suffers from exactly that same limitation but is nevertheless widely interpreted as if the society was in a stationary state.

Finally, the definition of $ED_{\theta\hat{a}}$ is such that each year prematurely lost is as bad as $1/\theta$ years spent in poverty. This trade-off between the relative costs of poverty and mortality is the same as for $PALE_{\theta}$.[18] When $\theta = 1$, $ED_{\theta\hat{a}}$ has a transparent interpretation, as it computes the expected proportion of the normative lifespan that a newborn expects to lose prematurely or spend in poverty.

Unlike $PALE_{\theta}$, $ED_{\theta\hat{a}}$ accounts for the distributional concern in the mortality dimension. The age threshold $\hat{a}$, above which some deaths are normatively irrelevant, is to mortality what the poverty line is to poverty. In Proposition 3, we show that index $ED_{\theta\hat{a}}$ is a generalization of index $PALE_{\theta}$: $ED_{\theta\hat{a}}$ ranks societies exactly in the same way as $PALE_{\theta}$ as long as its age threshold $\hat{a}$ is at least as large as the maximal age a^*. For such values, the age threshold becomes not binding, and all deaths become relevant in terms of deprivation because they all occur at younger ages than

[18] We assume here that the welfare cost of a year prematurely lost is equal to u_{NP}.

the age threshold. When the age treshold is binding (smaller than the maximal age a^*), the rankings obtained under $ED_{\theta\hat{a}}$ do not correspond to the rankings obtained under $PALE_\theta$.

Proposition 3 ($ED_{\theta\hat{a}}$ generalizes $PALE_\theta$)**.**
For all $\hat{a} \geq a^*$ we have $PALE_\theta = \hat{a}(1 - ED_{\theta\hat{a}})$, which implies that, for any two societies A and B,

$$PALE_\theta(A) \geq PALE_\theta(B) \Leftrightarrow ED_{\theta\hat{a}}(A) \leq ED_{\theta\hat{a}}(B).$$

Proof. See Appendix 5.3. □

Taken together, Propositions 2 and 3 show that $ED_{\theta\hat{a}}$ aggregates two indices of mortality, LE and $LGE_{\hat{a}}$, with an index of poverty, H, in a way which is consistent with life-cycle preferences. This improves on standard multidimensional poverty indices. However, $ED_{\theta\hat{a}}$ relies on two normative parameters: $(\theta, \hat{a})$. Proposition 4 below provides the conditions under which the ranking by $ED_{\theta\hat{a}}$ for *some* pairs of societies A and B does not depend on the values selected for its normative parameters.

Proposition 4 (Unambiguous comparisons of deprivation)**.**
(i) For any two societies A and B we have $ED_{\theta\hat{a}}(A) > ED_{\theta\hat{a}}(B)$ for all $\theta \in [0,1]$ and all $\hat{a} \geq \underline{\hat{a}}$ if and only if

$$\begin{aligned} &ED_{0\hat{a}}(A) > ED_{0\hat{a}}(B) \text{ for all } \hat{a} \geq \underline{\hat{a}}, \text{ and} \\ &ED_{1\hat{a}}(A) > ED_{1\hat{a}}(B) \text{ for all } \hat{a} \geq \underline{\hat{a}} \qquad \text{(generalized Condition C1)} \end{aligned}$$

(ii) For any $\underline{\hat{a}} > 1$, there exist societies A and B for which $ED_{\theta\hat{a}}(A) > ED_{\theta\hat{a}}(B)$ for all $\theta \in [0,1]$ and all $\hat{a} \geq \underline{\hat{a}}$ even though $H(A) < H(B)$. These societies are such that $LE(A) < LE(B)$.

Proof. See Appendix 5.4 for the straightforward proof. □

The first part of Proposition 4 tells us that $ED_{\theta\hat{a}}$ provides unambiguous comparisons if $ED_{0\hat{a}}$ and $ED_{1\hat{a}}$ provide the same ranking for all age thresholds $\hat{a}$ above $\underline{\hat{a}}$. The intuition for this result is essentially the same as that provided in Proposition 1 above (since, when $\hat{a} \geq a^*$, $ED_{\theta\hat{a}}$ is equivalent to $PALE_\theta$).

The second part of the Proposition indicates when ignoring mortality and focusing exclusively on H leads to deprivation comparisons that are unambiguously correct or wrong. When $\underline{\hat{a}} < a^*$, it no longer suffices that H and LE yield separately the same ranking for that ranking to be unambiguously correct. The reason is that, when $\underline{\hat{a}} < a^*$, LE no longer contains all the relevant information on mortality: for instance, two societies can share the same life-expectancy at birth but one with several deaths occuring below $\hat{a}$ while the other has all deaths occuring above $\hat{a}$. Note also that the larger the lower-bound $\underline{\hat{a}}$, i.e., the smaller the set of plausible values for $\hat{a}$, the larger the set of comparisons for which the generalized condition can be met.

We illustrate the above results in Figure 8.[19] The vertical axis represents the share of pairs of societies for which H and LE provide similar (at the top) or opposite rankings (at the bottom). By definition, these rankings are insensitive to the age

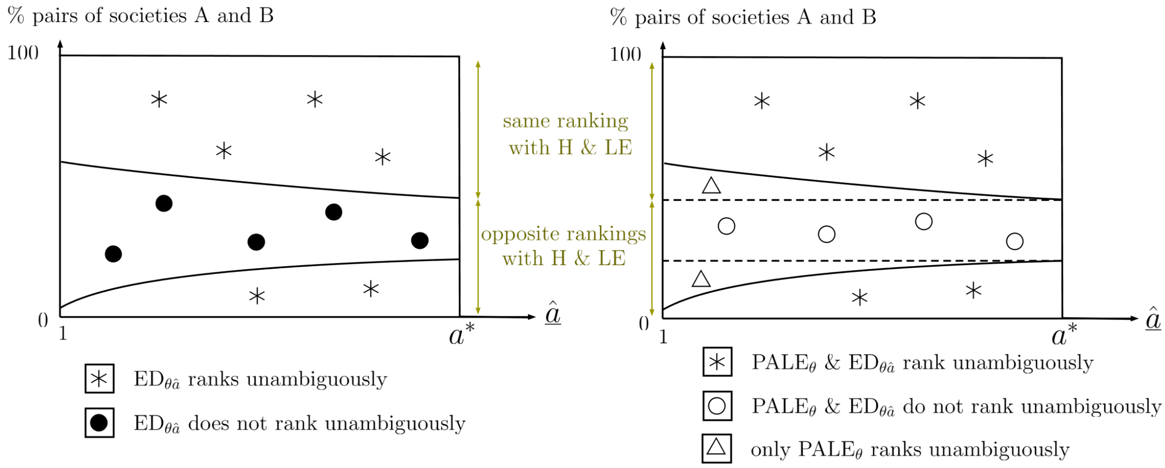
[19] All graphs that follow are constructed using a lower bound on $\hat{a}$ equal to 1. Indeed, for $\theta = 0$ and $\hat{a} = 0$, $ED_{\theta\hat{a}}$ is equal to zero for all societies and cannot therefore deliver unambiguous comparisons.

threshold $\hat{a}$ considered. The horizontal axis represents all possible values of $\underline{\hat{a}}$, the lower bound on the age threshold.

The left panel describes the share of cases in which $ED_{\theta\hat{a}}$ provides unambiguous rankings as a function of $\underline{\hat{a}}$. Lower values of $\underline{\hat{a}}$ imply a fall in the share of cases that $ED_{\theta\hat{a}}$ can rank unambiguously. Indeed, a larger age interval over which $ED_{\theta\hat{a}}$ has to be computed implies a larger number of comparisons for ED. As a result, the number of cases for which it can provide the same ranking for all age thresholds falls. Second, if H and LE agree, $ED_{\theta\hat{a}}$ provides the same ranking as H when $\underline{\hat{a}} = a^*$. Finally, as discussed above, when H and LE disagree, a larger value of $\underline{\hat{a}}$ implies that the share of cases for which H provides a unambiguously wrong ranking gets larger.

The right panel reports, for all values of $\underline{\hat{a}}$, the share of pairs of societies for which $PALE_\theta$ and $ED_{\theta\hat{a}}$ provide unambiguous rankings. Since $PALE_\theta$ doesn't depend on the age threshold, it is able to rank a larger set of comparisons. As shown in Proposition 3, when $\underline{\hat{a}} = a^*$, the two indices are equivalent.

Figure 8: H may make unambiguously wrong deprivation comparisons when $\underline{\hat{a}} > 1$.



Reading: The smaller the lower-bound $\underline{\hat{a}}$, the lower the share of societies pairs unambiguously ranked by $ED_{\theta\hat{a}}$ even when H and LE agree with one another. The higher the lower-bound $\underline{\hat{a}}$, the higher the share of societies pairs unambiguously ranked by both $ED_{\theta\hat{a}}$ and $PALE_\theta$.

3.2 Mortality shocks and the evolution of $ED_{\theta\hat{a}}$ and $PALE_\theta$

We now briefly contrast the impact of mortality shocks on $PALE_\theta$ and $ED_{\theta\hat{a}}$, assuming that these mortality shocks are independent of the poverty status. Consider a mortality shock that equalizes individual lifespans across the age threshold $\hat{a}$ while keeping life-expectancy LE constant. This lower dispersion in mortality does not affect $PALE_\theta$, which only accounts for mortality through LE. By contrast, this shock reduces $ED_{\theta\hat{a}}$, since $LGE_{\hat{a}}$ is thereby reduced. It is indeed easy to show that $\frac{\partial ED_{\theta\hat{a}}}{\partial LGE_{\hat{a}}} > 0$ (for $\theta H < 1$).

Consider instead a mortality shock that reduces mortality above the age threshold $\hat{a}$. Such shock increases LE but does not affect $LGE_{\hat{a}}$. As a result, $PALE_\theta$ mechanically increases. It is also easy to show that deprivation, as measured by $ED_{\theta\hat{a}}$, decreases: $\frac{\partial ED_{\theta\hat{a}}}{\partial LE} < 0$, for $\theta H < 1$. Moreover, $PALE_\theta$ is more sensitive to this kind of shock than $ED_{\theta\hat{a}}$, as the elasticity of $PALE_\theta$ to LE is equal to 1 while the elasticity of $ED_{\theta\hat{a}}$ to LE lies in $(-1, 0)$. If the mortality shock is such that it reduces mortality below the age threshold $\hat{a}$, this shock simultaneously increases LE and reduces $LGE_{\hat{a}}$. Again, $PALE_\theta$ improves and deprivation decreases since both LE increases and LGE decreases.

3.3 Empirical relation between $ED_{\theta\hat{a}}$ and $PALE_\theta$

Figure 9 reports the diagnostic delivered by $PALE_\theta$ and $ED_{\theta\hat{a}}$ over all pairwise comparisons of countries in 2019 for which H and LE yield opposite rankings, focusing on the share of these cases that can be ranked by $ED_{\theta\hat{a}}$ and $PALE_\theta$ independently of the value assigned to θ for a given age threshold.

Since $PALE_\theta$ does not depend on the age threshold, the discrepancies between $PALE_\theta$ and $ED_{\theta\hat{a}}$ across ages can only come from variations in $ED_{\theta\hat{a}}$. As is clear from the Figure, the share of cases solved by $PALE_\theta$ is constant but the share of cases solved by both $PALE_\theta$ and $ED_{\theta\hat{a}}$ increases with the age threshold. Since the age threshold acts as a form of weight on the poverty component of $ED_{\theta\hat{a}}$, the relative importance of poverty in $ED_{\theta\hat{a}}$ decreases as $\hat{a}$ increases. When $\hat{a} \geq a^*$, mortality and poverty have the same weight in $ED_{\theta\hat{a}}$ and $PALE_\theta$ and the two indices yield exactly the same ranking (Proposition 3). For lower age thresholds, the number of periods of life considered as prematurely lost decreases, and $ED_{\theta\hat{a}}$ becomes more sensitive to its poverty component. In the extreme case in which $\hat{a} = 0$, $ED_{\theta 0}$ can solve all the cases for which H and LE yield opposite rankings since $ED_{\theta 0}$ is unidimensional and equal to θH.

Figure 9: Resolution of ambiguous comparisons of $PALE_\theta$ and $ED_{\theta\hat{a}}$, by age threshold

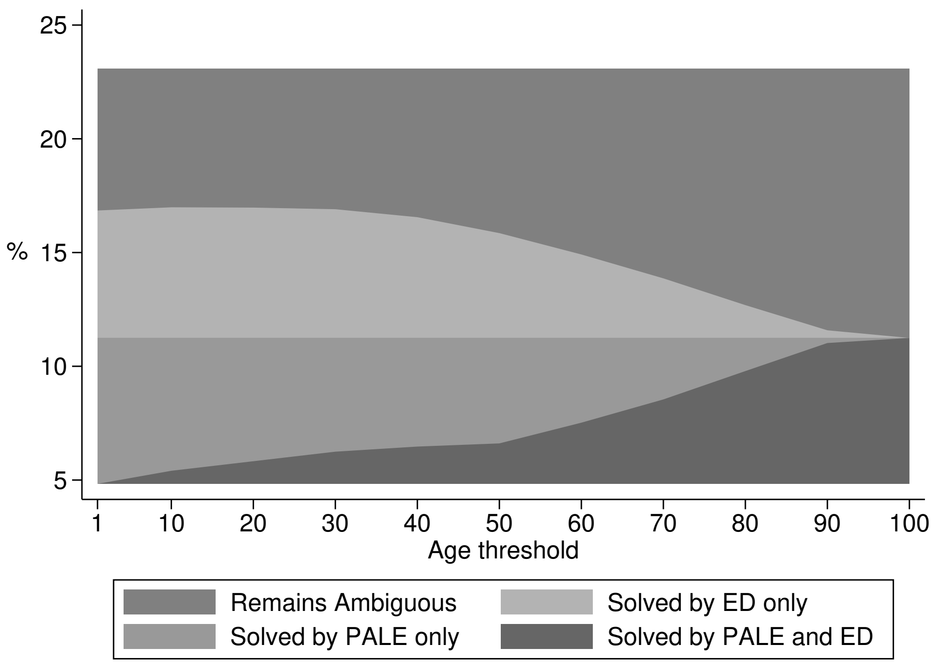
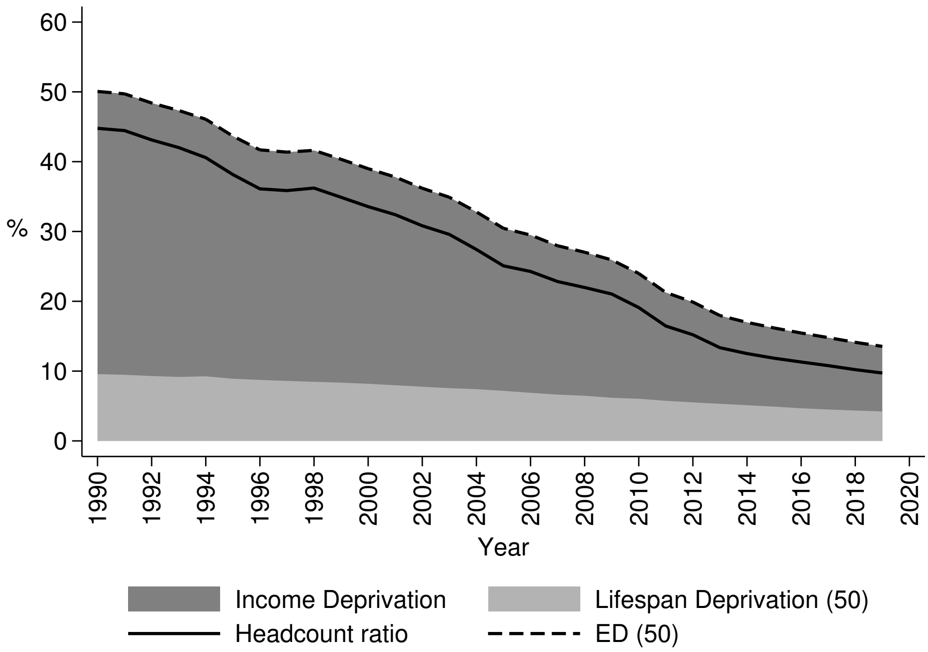


Figure 10 presents the evolution of $ED_{\theta\hat{a}}$ (with $\theta = 1$ and $\hat{a} = 50$) and the headcount ratio, H, for the world. As can be seen from the figure, the incidence of poverty massively decreased over that period, while premature mortality (that is, before the age 50) decreased at a much lower rate. As a result, at the world scale, $ED_{1,50}$ follows closely the evolution of H and gets closer in the recent years. Overall, in 2019, a newborn can expect, under stationarity, to lose 15% of his normative lifespan in poverty of through premature mortality. The corresponding figure in 1990 was as high as 50%.

Figure 10: Evolution of $ED_{\theta\hat{a}}$ and H, 1990-2019 (where $\theta = 1$ and $\hat{a} = 50$).



3.4 Relation with other indices of deprivation

We limit here our comparison to other indices in the literature to the index of generated deprivation ($GD_{\theta\hat{a}}$) we proposed in a companion paper (Baland et al., 2021). Generated deprivation is indeed the closest index to $ED_{\theta\hat{a}}$, and Baland et al. (2021) discuss in details the relationships between $GD_{\theta\hat{a}}$ and other indices of multidimensional poverty. In short, $ED_{\theta\hat{a}}$ and $GD_{\theta\hat{a}}$ are identical in stationary societies, but $ED_{\theta\hat{a}}$ has a simpler interpretation than $GD_{\theta\hat{a}}$, reacts faster to permanent mortality shocks and is less demanding in terms of data. However, $GD_{\theta\hat{a}}$ is decomposable in subgroups whereas $ED_{\theta\hat{a}}$ is not.

In any year t, the $GD_{\theta\hat{a}}$ index is defined as follows: [20]

$$GD_{\theta\hat{a}} = \underbrace{\frac{YL_t}{N_t + YL_t}}_{quantity\ deprivation} + \theta \underbrace{\frac{N_t * H_t}{N_t + YL_t}}_{quality\ deprivation}, \tag{6}$$

[20]Strictly speaking, the generated deprivation index proposed in Baland et al. (2021) is $\frac{1}{\theta} GD_{\theta\hat{a}}$, which is ordinally equivalent to $GD_{\theta\hat{a}}$ since θ is a constant.

where $\theta \in [0,1]$ and $N_t = \sum_{a=0}^{a^*-1} n_{t-a}(a)$ is the population observed in t. YL_t is the total number of years of life prematurely lost due to mortality in year t, and can be defined as follows:

$$YL_t = \sum_{a=0}^{\hat{a}-2} n_{t-a}(a) * \mu_a^t * (\hat{a} - (a+1)),$$

where $\mu^t = (\mu_0^t, \dots, \mu_{a^*-1}^t)$ stands for the vector of age-specific mortality rates.[21]

The $GD_{\theta\hat{a}}$ index is also based on two components, one capturing quality deprivation, measured by the number of man-years spent in poverty in year t, and the other quantity deprivation, measured by all the years prematurely lost in year t. When compared to $ED_{\theta\hat{a}}$, the same normative weight is also used for these two components. The components of $GD_{\theta\hat{a}}$ are however harder to interpret. This is because $GD_{\theta\hat{a}}$ combines a number of poor with a number of years of life prematurely lost. The rationale behind this aggregation is that, in a given year, the total number of "poor individuals" in a given year also corresponds to the total number of "years" lived in poverty in that year. This equivalence also explains why the denominator of $GD_{\theta\hat{a}}$ sums a number of individuals, N_t with a number of years YL_t. By contrast, the numerators of the two terms in $ED_{\theta\hat{a}}$ are more easily interpretable: they are the number of years that a newborn expects to prematurely lose or spend in poverty (if she expects mortality and poverty to stay at their currently observed levels).

The following proposition establishes that $GD_{\theta\hat{a}}$ and $ED_{\theta\hat{a}}$ are identical in stationary societies:

Proposition 5 ($ED_{\theta\hat{a}}$ and $GD_{\theta\hat{a}}$ are identical in stationary societies)**.**
For any stationary society,

$$LGE_{\hat{a}} = \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \mu_a^t * \prod_{k=0}^{a-1} (1 - \mu_k^t), \tag{7}$$

which yields $GD_{\theta\hat{a}} = ED_{\theta\hat{a}}$.

Proof. See Appendix 5.5. □

$ED_{\theta\hat{a}}$ and $GD_{\theta\hat{a}}$ yield the same ranking for stationary societies. However, societies are typically not stationary so that $ED_{\theta\hat{a}}$ and $GD_{\theta\hat{a}}$ may rank countries differently. The main difference between $ED_{\theta\hat{a}}$ and $GD_{\theta\hat{a}}$ comes from the way the two indices compute the number of years prematurely lost. $ED_{\theta\hat{a}}$ takes the perspective of a newborn who faces throughout her life the mortality rates observed in t. In contrast, $GD_{\theta\hat{a}}$ computes the number of years that are lost by the current population due to the premature mortality observed in t. It records, over all premature deaths in t, the number of years prematurely lost. Thus, if an individual dies at age 20 and the age threshold is 70, her premature death leads to a loss of 50 years of life in that year. $ED_{\theta\hat{a}}$ also counts the number of years prematurely lost, but instead of being computed on the actual population pyramid, $ED_{\theta\hat{a}}$ uses a counterfactual population pyramid, which is the one that would prevail in a stationary society characterized by the age-specific mortality rates observed in the period.

[21] They are more formally defined in Appendix 5.2.

A major implication of this difference is that $ED_{\theta\hat{a}}$ is more reactive to policy changes than $GD_{\theta\hat{a}}$. Consider a permanent mortality shock. The population dynamics is such that a transition phase sets in during which the population pyramid slowly adjusts to the new mortality rates. This transition stops when a new stationary population pyramid is reached, typically after a^* periods. $GD_{\theta\hat{a}}$ records each step of this transition and therefore exhibits inertia in its response to a permanent mortality shock. By contrast, $ED_{\theta\hat{a}}$ immediately refers to the new stationary population pyramid and disregards the inertia caused by these transitory demographic adjustments. We provide an illustration of this property in Appendix 6.

Finally, Baland et al. (2021) show that $GD_{\theta\hat{a}}$ is essentially the only index decomposable in subgroups to compare stationary societies in a way that satisfy some basic properties. As a result, $ED_{\theta\hat{a}}$ cannot be decomposable in subgroups.[22] This is no surprise given that $ED_{\theta\hat{a}}$ is based on life expectancy, which cannot be decomposed in subgroups. In Appendix 7, we also show that $ED_{\theta\hat{a}}$ is essentially the only index that is independent on the actual population pyramid and compares stationary population in a way that respect basic properties of deprivation. As a result, the actual population pyramid is irrelevant for $ED_{\theta\hat{a}}$, the only information required for $ED_{\theta\hat{a}}$ is age-specific mortality rates.

4 Concluding remarks

An important limitation of the two indices proposed in this paper, $PALE_\theta$ and $ED_{\theta\hat{a}}$, is that they account for the distributional concern "dimension-by-dimension" instead of accounting for them in terms of life-cycle utility. Indeed, our indices are insensitive to the allocation of years of life prematurely lost between the poor and the non-poor. This allocation may however have important implications for the distribution of life-cycle utility. Indeed, when the poor die early, they cumulate low achievements in the two dimensions and the difference between their life-cycle utility and that of the non-poor increases.

Without denying the importance of this limitation, let us first note that this limitation is shared by most standard indices of human development.[23] Second, addressing this limitation requires data that are typically not available. One natural way of accounting for such "concentration" of deprivations on the same individuals would be to define as "life-cycle poor" individuals whose life-cycle utility is smaller than that of a reference life, e.g., a life characterized by a lifespan of 40 years with no period of poverty. One can then define an index of human development that would, for instance, correspond to the expected fraction of newborns who will be "life-cycle poor". This type of index would not be ad-hoc, but would require better data, combining poverty and mortality at the individual level, than what is currently available in most countries. Moreover, this type of data, recording mortality up to

[22] In other words, if decomposibility in subgroups is seen as a key property, one should use $GD_{\theta\hat{a}}$. Indeed, this index yields the same ranking as $ED_{\theta\hat{a}}$ in stationary populations. In those populations, $GD_{\theta\hat{a}}$ thus yields the same ranking as $PALE_\theta$ when all deaths are normatively relevant ($\hat{a} \geq a^*$).

[23] To the best of our knowledge, the global MPI index is the only one to account, in an indirect way, for such concentration. In a nutshell, the deprivation-score of an individual is increased if she lives in a household that has experienced the death of a less than 18 year child in the past five years. Arguably, this aggregation of the quantity and quality of life is essentially practical. It is not related to any concept of life-cycle utility. Moreover, it critically depends on the definition of the household as it does not account for the occurrence of multiple deaths in the same households.

a given threshold, would necessarily be historical in nature, with little relevance to the current situation. Alternatively, one may want to define indices that are less demanding in terms of information, and are based on the observed mobility between poverty and non-poverty, as well as on mortality figures for the poor and the non-poor. Some additional assumptions would be required to then translate this information into the lifecycle profiles for newborns.

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5 Appendix: Proofs

5.1 Proof of Proposition 1

Proof of (i). We start by the "only if" part. Assume to the contrary that $PALE_0(A) > PALE_0(B)$ or $PALE_1(A) > PALE_1(B)$. This directly implies that $PALE_\theta(A) > PALE_\theta(B)$ for some $\theta \in \{0, 1\}$ and therefore we cannot have $PALE_\theta(A) < PALE_\theta(B)$ for all $\theta \in [0, 1]$.

We now turn to the "if" part. By definition of the PALE$_\theta$ index, we have to show that

$$LE(B) - LE(A) > \theta * (H(B) - H(A)), \tag{8}$$

for all $\theta \in [0, 1]$. As $PALE_0(A) < PALE_0(B)$, we directly have that $LE(B) - LE(A) > 0$ because $PALE_0 = LE$. As $PALE_1(A) < PALE_1(B)$, we have $LE(B) - LE(A) > H(B) - H(A)$. It immediately follows that the inequality (8) is verified for all values of θ smaller than 1.

Proof of (ii). From (i), proving (ii) only requires providing societies A and B with $H(A) < H(B)$ such that $PALE_0(A) < PALE_0(B)$ and $PALE_1(A) < PALE_1(B)$. If $H(A) = 0.2$, $H(B) = 0.4$, $LE(A) = 50$ and $LE(B) = 75$ we have $PALE_1(A) = 40$ and $PALE_1(A) = 45$, the desired result because $PALE_0 = LE$.

5.2 Stationary societies and PALE$_\theta$

We first provide a formal definition of a stationary society. Consider a discrete set of periods $\{\ldots, t-1, t, t+1, \ldots\}$. In each period, some individuals are born and some individuals die (at the end of the period). All alive individuals are assigned a consumption status for the period (P or NP) . We define the life of an individual i as the list of consumption statuses $l_i = (l_{i0}, \ldots, l_{id_i})$ she enjoys between age 0 and age $d_i \in \{0, \ldots, a^* - 1\}$ at which she dies, where $l_{ia} \in \{NP, P\}$. The set of lives is thus $L = \cup_{d \in \{0,\ldots,a^*-1\}} \{NP, P\}^{d+1}$.

The number of newborns in period t is denoted by n_t. The profile of lives for the cohort born in t is denoted by $C_t = (l_i)_{i \in \{1,\ldots,n_t\}}$, where $\{1, \ldots, n_t\}$ is the set of newborns in t. Clearly, the profile of lives C_t contains all the information necessary to compute a newborn's expected life-cycle utility (Eq. (3)). Let $n_t(a)$ denote the number of individuals born in period t who are still alive when reaching age a. In particular, we have $n_t(0) = n_t$. Let $p_t(a)$ denote the number of individuals born in period t who are poor at age a, with $p_t(a) \leq n_t(a)$. By definition, the probability that an individual born in t survives to age a is given by $V_t(a) = \frac{n_t(a)}{n_t}$, and the conditional probability that an individual born in t will be poor when reaching age a is $\pi_t(a) = \frac{p_t(a)}{n_t(a)}$. To compute Eq. (3), it is sufficient to know the distribution of the set of lives that C_t implicitly defines. We denote this distribution by $\Gamma_t : L \to [0, 1]$, with $\sum_{l \in L} \Gamma_t(l) = 1$.

In period t, we cannot observe the profile of lives for the cohort born in t. The only elements of C_t that we observe in that period are $n_t(0)$, $p_t(0)$ and $n_t(1)$. However, we also have information about the profile of lives of the cohorts born *before* t. Formally, let a **society** S_t be the list of profiles of lives for all cohorts born during the a^* periods in $\{t-(a^*-1),\dots,t\}$, i.e. $S_t = (C_{t-a^*+1},\dots,C_t)$. In period t, we observe (i) the number N_t of individuals who are alive in t:

$$N_t = \sum_{a=0}^{a^*-1} n_{t-a}(a),$$

(ii) the fraction H_t of alive individuals who are poor in t:

$$H_t = \frac{\sum_{a=0}^{a^*-1} p_{t-a}(a)}{\sum_{a=0}^{a^*-1} n_{t-a}(a)},$$

and (iii) the age-specific mortality vector $\mu^t = (\mu_0^t,\dots,\mu_{a*-1}^t)$ in period t where for each $a \in \{0,\dots,a^*-1\}$ we have

$$\mu_a^t = \frac{n_{t-a}(a) - n_{t-a}(a+1)}{n_{t-a}(a)},$$

with $\mu_{a^*-1}^t = 1$.

We now show that, in stationary societies, the information available in period t is sufficient to compute the value of the expected lifecycle utility using Eq. (3). The particularity of stationary societies is to have their natality, mortality and poverty constant over time, so that all (average) outcomes in a given period are replicated over the next period. More formally, a society is stationary if both the distribution of lives and the size of generations are constant over the last a^* periods.

Definition 1 (Stationary Society)**.**
A society S_t is stationary if, at any period $t' \in \{t-a^*+1,\dots,t\}$, we have

- $\Gamma_{t'} = \Gamma_t$ *(constant distribution of lives),*
- $n_{t'} = n_t$ *(constant size of cohorts).*

It follows from this definition that $n_t(a) = n_{t-a}(a)$ and $p_t(a) = p_{t-a}(a)$ for all $a \in \{1,\dots,a^*-1\}$.[24] These equalities lead to the following Lemma, which allows us to relate Eq. (3) to the information available in period t.[25]

Lemma 1. *If S_t is stationary,*

$$V_t(a) = \Pi_{k=0}^{a-1}(1-\mu_k^t) \qquad \textit{for all } a \in \{0,\dots,a^*-1\}, \tag{9}$$

$$N_t = n_t * LE_t, \tag{10}$$

$$N_t * H_t = n_t * \sum_{a=0}^{a^*-1} V(a)\pi(a). \tag{11}$$

Proof. We first prove Eq (9). As S_t is stationary, we have $n_t(k) = n_{t-k}(k)$ for all $k \in \{1,\dots,a^*-1\}$ and $n_t(k+1) = n_{t-k}(k+1)$ for all $k \in \{0,\dots,a^*-2\}$. Therefore,

[24] Clearly, a constant distribution of lives is not sufficient for these equalities, one also needs a constant size of cohorts.

[25] Lemma 1 also requires that $n_t(a+1) = n_{t-a}(a+1)$ for all $a \in \{0,\dots,a^*-2\}$, which follows from the definition of a stationary society.

we have for all $a \in \{1, \ldots, a^* - 1\}$ that

$$\begin{aligned} V_t(a) &= \frac{n_t(a)}{n_t}, \\ &= \Pi_{k=0}^{a-1} \frac{n_t(k+1)}{n_t(k)}, \\ &= \Pi_{k=0}^{a-1} \frac{n_{t-k}(k+1)}{n_{t-k}(k)}, \\ &= \Pi_{k=0}^{a-1} (1 - \mu_k^t). \end{aligned}$$

We then prove Eq (10). As S_t is stationary, we have $n_t(a) = n_{t-a}(a)$ for all $a \in \{1, \ldots, a^* - 1\}$. Recalling that $V_t(a) = \frac{n_t(a)}{n_t}$, we can successively write

$$\begin{aligned} LE_t &= \sum_{a=0}^{a^*-1} V_t(a), \\ &= \frac{\sum_{a=0}^{a^*-1} n_t(a)}{n_t}, \\ &= \frac{\sum_{a=0}^{a^*-1} n_{t-a}(a)}{n_t}, \\ &= N_t / n_t. \end{aligned}$$

Finally, we prove Eq. (11). As S_t is stationary, we have $p_t(a) = p_{t-a}(a)$ for all $a \in \{1, \ldots, a^* - 1\}$. Given that $\pi_t(a) = \frac{p_t(a)}{n_t(a)}$ and $V_t(a) = \frac{n_t(a)}{n_t}$, we can successively write

$$\begin{aligned} H_t &= \frac{\sum_{a=0}^{a^*-1} p_{t-a}(a)}{\sum_{a=0}^{a^*-1} n_{t-a}(a)}, \\ &= \frac{\sum_{a=0}^{a^*-1} p_t(a)}{N_t}, \\ &= \frac{\sum_{a=0}^{a^*-1} \pi_t(a) V_t(a) n_t}{N_t}. \end{aligned}$$

□

The three equations in Lemma 1 imply that an individual born in a stationary society can infer her expected life-cycle utility from the information available at the year of her birth. (These direct relationships between current and future outcomes in stationary societies are well-known to demographers (Preston et al., 2000).) We illustrate this important insight using an example. Consider a stationary society for which two individuals are born in each cohort, one living only for one period in poverty and the other living for two periods out of poverty, i.e. $n = 2$, $l_1 = (P)$ and $l_2 = (NP, NP)$. In period t, three individuals are alive: the poor born in t, the non-poor born in t and the non-poor born in $t-1$. Also, two individuals die at the end of period t: the poor born in t and the non-poor born in $t-1$. Eq. (9) states that the mortality rates observed in period t (the right hand side of the equation) can be used to infer the mortality rates that the newborn can expect to face during her life-cycle (the left hand side). Thus, in our example, a newborn observes that, at the end of period t, half of the individuals of age 0 die and all individuals of age 1 die. Eq. (9) implies that he has a 50 percent chance to survive period t and a zero percent chance to survive period $t+1$. According to Eq. (10), the number of individuals who are alive in period t, N_t, is equal to the number of person-periods in the profile of lives of the cohort born in period t. In our example, there are three individuals alive in period t and there are three person-periods in $C_t = (l_1, l_2) = (P; NP, NP)$. Finally, Eq. (11) states that the number of poor observed in period t, $N_t * H_t$, is equal to the number of person-periods of poverty in the profile of lives of the cohort born in period t. Indeed, there is one poor individual alive in period t and one person-period P in C_t.

Lemma 1 shows that, in a stationarity society, the poverty and mortality observed in a given period perfectly define the life profile of newborns. Proposition 6 shows that $PALE_\theta$ is a normalisation of the expected life-cycle utility of a newborn in a stationary society even when mortality is selective, i.e. when the conditional probability of being poor depends on age.

Proposition 6 (Equivalence between Harsanyi and $PALE_\theta$).
If society S_t is stationary, then $PALE_\theta = \frac{EU_t}{u_{NP}}$.

Proof. The result follows directly when substituting Eq. (10) and (11) into Eq. (3). ☐

5.3 Proof of Proposition 3

The proof builds on the complete framework presented in Appendix 5.2.

We first show that $LE + LGE_{\hat{a}} = \hat{a}$ when $\hat{a} \geq a^*$.

$$\begin{aligned} LGE_{\hat{a}}(C_t) &= \sum_{a=0}^{\hat{a}-1} \hat{a} * \frac{n_t(a) - n_t(a+1)}{n_t} - \sum_{a=0}^{\hat{a}-1} (a+1) * \frac{n_t(a) - n_t(a+1)}{n_t}, \\ &= \frac{1}{n_t}\left(\hat{a} * (n_t(0) - n_t(\hat{a})) - \sum_{a=0}^{\hat{a}-1} n_t(a) + \hat{a} * n_t(\hat{a}) \right), \\ &= \hat{a} - \sum_{a=0}^{\hat{a}-1} \frac{n_t(a)}{n_t}. \end{aligned}$$

By definition of a^*, we have $n_t(a) = 0$ for all $a \geq a^*$. When $\hat{a} \geq a^*$, this implies that $\sum_{a=0}^{\hat{a}-1} \frac{n_t(a)}{n_t} = \sum_{a=0}^{a^*-1} \frac{n_t(a)}{n_t}$, where by definition $LE = \sum_{a=0}^{a^*-1} \frac{n_t(a)}{n_t}$, the desired result.

The fact that $LE + LGE_{\hat{a}} = \hat{a}$ implies that $PALE_\theta = \hat{a}(1 - ED_{\theta\hat{a}})$ because

$$\hat{a}(1 - ED_{\theta\hat{a}}) = (LE + LGE_{\hat{a}})(1 - ED_{\theta\hat{a}}) = LE(1 - \theta H).$$

Thus, when $\hat{a} \geq a^*$, $PALE_\theta$ is a linear function of $ED_{\theta\hat{a}}$ that depends negatively on $ED_{\theta\hat{a}}$. Therefore, these two indicators yields opposite ranking of any two societies A and B, i.e. $PALE_\theta(A) \geq PALE_\theta(B) \Leftrightarrow ED_{\theta\hat{a}}(A) \leq ED_{\theta\hat{a}}(B)$.

5.4 Proof of Proposition 4

We first prove the following: for any $\hat{a} \geq \underline{\hat{a}}$ and any two societies A and B, we have $ED_{\theta\hat{a}}(A) > ED_{\theta\hat{a}}(B)$ for all $\theta \in [0, 1]$ if and only if

$$ED_{0\hat{a}}(A) > ED_{0\hat{a}}(B) \text{ and } ED_{1\hat{a}}(A) > ED_{1\hat{a}}(B).$$

We start with the "only if" part. Assume on the contrary that $ED_{0\hat{a}}(A) < ED_{0\hat{a}}(B)$ or $ED_{1\hat{a}}(A) < ED_{1\hat{a}}(B)$. This implies that $ED_{\theta\hat{a}}(A) < ED_{\theta\hat{a}}(B)$ for some $\theta \in \{0,1\}$ and therefore we cannot have $ED_{\theta\hat{a}}(A) > ED_{\theta\hat{a}}(B)$ for all $\theta \in [0,1]$.

We turn to the "if" part. By definition of the $ED_{\theta\hat{a}}$ index, we have to show that

$$\frac{LGE_{\hat{a}}(A)}{LE(A)+LGE_{\hat{a}}(A)} - \frac{LGE_{\hat{a}}(B)}{LE(B)+LGE_{\hat{a}}(B)} > \theta\left(\frac{LE(B)*H(B)}{LE(B)+LGE_{\hat{a}}(B)} - \frac{LE(A)*H(A)}{LE(A)+LGE_{\hat{a}}(A)}\right) \text{ for all } \theta \in [0,1]. \tag{12}$$

As $ED_{1\hat{a}}(A) > ED_{1\hat{a}}(B)$, Eq. (12) holds for $\theta = 1$. As $ED_{0\hat{a}}(A) > ED_{0\hat{a}}(B)$, the left hand side of Eq. (12) is strictly positive. As a result, inequality (12) holds for all values of θ smaller than 1.

Proof of (i). This is an immediate implication of the statement proven above.

Proof of (ii). Consider two societies A and B with $H(A) < H(B)$ for which the generalized condition C1 holds.

Society A is such that $H(A) = 0.4$ and all its individuals die in their first year of life, which implies that $LE(A) = 1$ and $LGE_{\hat{a}}(A) = \hat{a} - 1$. Therefore, society A is such that $ED_{0\hat{a}}(A) = \frac{\hat{a}-1}{\hat{a}}$ and $ED_{1\hat{a}}(A) = 1 - \frac{0.6}{\hat{a}}$ for all $\hat{a} \geq \underline{\hat{a}}$. Society B is such that $H(B) = 0.5$ and all its individuals die at the maximal age a^*, which implies that $LE(B) = a^* + 1$ and $LGE_{\hat{a}}(B) = 0$. Therefore, society B is such that $ED_{0\hat{a}}(B) = 0$ and $ED_{1\hat{a}}(B) = 0.5$ for all $\hat{a} \in \{2, \ldots, a^*\}$.

By the statement we have proven above, we have $ED_{\theta\hat{a}}(A) > ED_{\theta\hat{a}}(B)$ for all $\theta \in [0,1]$ and all $\hat{a} \in \{2, \ldots, a^*\}$ because

$$ED_{0\hat{a}}(A) > ED_{0\hat{a}}(B) \text{ for all } \hat{a} \in \{2, \ldots, a^*\}$$

as $\frac{\hat{a}-1}{\hat{a}} > 0$ for all $\hat{a} \in \{2, \ldots, a^*\}$, and

$$ED_{1\hat{a}}(A) > ED_{1\hat{a}}(B) \text{ for all } \hat{a} \in \{2, \ldots, a^*\}$$

as $\hat{a} > 1$ for all $\hat{a} \in \{2, \ldots, a^*\}$.

By (i), there remains to show that $ED_{\theta\hat{a}}(A) > ED_{\theta\hat{a}}(B)$ for all $\theta \in [0,1]$ and all $\hat{a} > a^*$. We have shown that $ED_{\theta a^*}(A) > ED_{\theta a^*}(B)$ for all $\theta \in [0,1]$, which implies by Proposition 3 that $PALE_\theta(A) < PALE_\theta(B)$ for all $\theta \in [0,1]$. By Proposition 3 again, $PALE_\theta(A) < PALE_\theta(B)$ for all $\theta \in [0,1]$ implies that $ED_{\theta\hat{a}}(A) > ED_{\theta\hat{a}}(B)$ for all $\theta \in [0,1]$ and all $\hat{a} > a^*$, the desired result.

5.5 Proof of Proposition 5

The proof builds on the complete framework presented in Appendix 5.2.

We first derive expression (7). As society S_t is stationary, we have that $n_t(a) = n_{t-a}(a)$ and $n_t(a+1) = n_{t-a}(a+1)$ for all $a \in \{0, \ldots, a^* - 1\}$. We can thus successively write

$$LGE_{\hat{a}}(S_t) = \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \frac{n_t(a) - n_t(a+1)}{n_t(a)} * \frac{n_t(a)}{n_t},$$
$$= \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \frac{n_{t-a}(a) - n_{t-a}(a+1)}{n_{t-a}(a)} * V_t(a).$$

As society S_t is stationary, Lemma 1 applies and we have $V_t(a) = \Pi_{k=0}^{a-1}(1 - \mu_k^t)$ (Eq. (9)). This result follows from our definition of the age-specific mortality rate, where $\mu_a^t = \frac{n_{t-a}(a) - n_{t-a}(a+1)}{n_{t-a}(a)}$.

We now prove that $GD_{\theta\hat{a}}(S_t) = ED_{\theta\hat{a}}(S_t)$. As society S_t is stationary, Lemma 1 applies and $N_t = n_t LE_t$ (Eq. (10)). Substituting this expression for N_t into the definition of $GD_{\theta\hat{a}}$ proves our result, provided $YL_t = n_t LGE_{\hat{a}}$, which remains to be shown. As society S_t is stationary, Lemma 1 applies and we have $\frac{n_{t-a}(a)}{n_t} = \prod_{k=0}^{a-1}(1 - \mu_k^t)$ (Eq. (9)). Substituting this expression for $n_{t-a}(a)$ into the definition of YL_t, with $YL_t = \sum_{a=0}^{\hat{a}-2} n_{t-a}(a) * \mu_a^t * (\hat{a} - (a+1))$, gives:

$$YL_t = n_t \sum_{a=0}^{\hat{a}-2} (\hat{a} - (a+1)) * \mu_a^t * \prod_{k=0}^{a-1} (1 - \mu_k^t),$$

which shows that $YL_t = n_t LGE_{\hat{a}}$ (see Eq. (7) and recall that $\hat{a} - (a+1) = 0$ when $a = \hat{a} - 1$), the desired result.

6 Appendix: $ED_{\theta\hat{a}}$ and $GD_{\theta\hat{a}}$ under a transitory shock: an illustration

We illustrate this difference between $ED_{\theta\hat{a}}$ and $GD_{\theta\hat{a}}$ in their reaction to a transitory mortality shock with the help of a simple example. Consider a population with a fixed natality $n_t(0) = 2$ for all periods t. At each period, all alive individuals are non-poor, implying that $H_t = 0$. For all $t < 0$, we assume a constant mortality vector $\mu^t = \mu^* = (0, 1, 1, 1)$, so that each individual lives exactly two periods. Let us assume $\hat{a} = 4$, so that an individual dies prematurely if she dies before her fourth period of life. Before period $t = 0$, the population pyramid is stationary, and the two indices are equal to $1/2$ because there is no poor and individuals live for two periods instead of four. Consider now a permanent shock starting from period 0 onwards, such that half of the newborns die after their first period of life: $\mu^0 = (1/2, 1, 1, 1)$. The population pyramid returns to its stationary state in period 1, after a (mechanical) transition in period 0. This example is illustrated in Figure 11.

Consider first $GD_{\theta\hat{a}}$. In period 0, the actual population pyramid is not stationary because of the mortality shock. The premature death of one newborn leads to the loss of three years of life. Also, two one-year old individuals die in period 0, each losing two years of life. There are thus 7 years of life prematurely lost in period 0, and $GD_{\theta\hat{a}}$ takes value $7/11$. In period 1, the population pyramid is stationary, and $GD_{\theta\hat{a}}$ is equal to $5/8$ from then on.

We now turn to $ED_{\theta\hat{a}}$. Even if the actual population pyramid is not stationary in period 0, $ED_{\theta\hat{a}}$ is immediately equal to $5/8$ since it records premature mortality

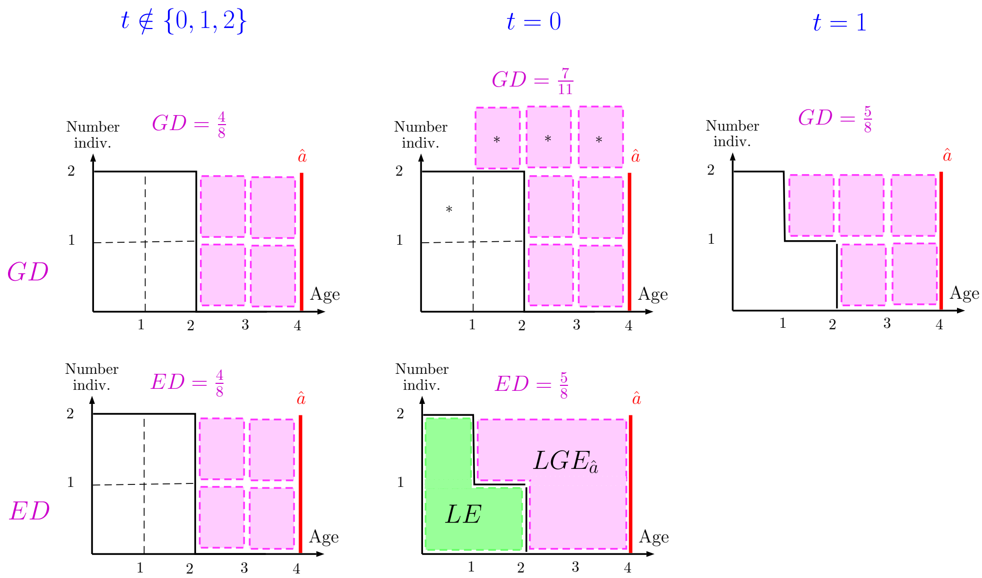

Figure 11: Response of $GD_{\theta\hat{a}}$ and $ED_{\theta\hat{a}}$ to a permanent mortality shock in $t = 0$. The years prematurely lost are shaded.

as if the population pyramid had already reached its new stationary level. $ED_{\theta\hat{a}}$ focusses on the newborn and the one-year old who die prematurely, ignoring that there are two one-year old dying in the actual population pyramid in period 0 (which is a legacy of the past).

7 Appendix: Characterization of the $ED_{\theta\hat{a}}$ index

We first introduce the set-up provided by Baland et al. (2021), which we will use to charcterize $ED_{\theta\hat{a}}$.

Each individual i is associated to a birth year $b_i \in \mathbb{Z}$. In period t, each individual i with $b_i \leq t$ is characterized by a **bundle** $x_i = (a_i, s_i)$, where $a_i = t - b_i$ is the age that individual i would have in period t given her birth year b_i, and s_i is a categorical variable capturing individual status in period t, which can be either alive and non-poor (NP), alive and poor (AP) or dead (D), i.e. $s_i \in S = \{NP, AP, D\}$. In the following, we often refer to individuals whose status is AP as "poor". We consider here that births occur at the beginning while deaths occur at the end of a period. As a result, an individual whose status in period t is D died before period t.[26]

An individual "dies prematurely" if she dies before reaching the minimal lifespan $\hat{a} \in \mathbb{N}$. Formally, period t is "prematurely lost" by any individual i with $s_i = D$ and $a_i < \hat{a}$. A **distribution** $x = (x_1, \ldots, x_{n(x)})$ specifies the age and the status in period t of all $n(x)$ individuals. Excluding trivial distributions for which no individual is alive or prematurely dead, the set of distributions in period t is given by:

$$X = \{x \in \cup_{n \in \mathbb{N}} (\mathbb{Z} \times S)^n \mid \text{there is } i \text{ for whom either } s_i \neq D \text{ or } s_i = D \text{ and } \hat{a} > t - b_i\}.$$

Baland et al. (2021) show that the most natural *consistent* index to rank distributions in X is the inherited deprivation index ($ID_{\theta\hat{a}}$). Let $d(x)$ denote the number

[26] All newborns have age 0 during period t and some among these newborns may die at the end of period t. This implies that $b_i = t \Rightarrow s_i \neq D$.

of *prematurely dead* individuals in distribution x, which is the number of individuals i for whom $s_i = D$ and $\hat{a} > t - b_i$, $p(x)$ the number of individuals who are poor and $f(x)$ the number of alive and non-poor individuals. The $ID_{\theta\hat{a}}$ index is defined as:

$$ID_{\theta\hat{a}}(x) = \underbrace{\frac{d(x)}{f(x)+p(x)+d(x)}}_{quantity\ deprivation} + \theta \underbrace{\frac{p(x)}{f(x)+p(x)+d(x)}}_{quality\ deprivation}, \tag{13}$$

where $\theta \in [0,1]$ is a parameter weighing the relative importance of alive deprivation and lifespan deprivation. An individual losing prematurely period t matters $1/\theta$ times as much as an individual spending period t in alive deprivation.

We introduce additional notation for the mortality taking place in period t. Consider the population pyramid in period t, and let $n_a(x)$ be the number of *alive* individuals of age a in distribution x, i.e. the number of individuals i for whom $a_i = a$ and $s_i \neq D$. (The definition of $n_a(x)$ corresponds to $n_{t-a}(a)$ in the notation used in the main text of the paper. In this section, we adopt the notation of Baland et al. (2021), which does not require to mention period t.) The age-specific mortality rate $\mu_a \in [0,1]$ denotes the fraction of alive individuals of age a dying at the end of period t: the number of a-year-old individuals dying at the end of period t is $n_a(x) * \mu_a$. Letting $a^* \in \mathbb{N}$ stand for the maximal lifespan (which implies $\mu_{a^*-1} = 1$), the **vector of age-specific mortality rates** in period t is given by $\mu = (\mu_0, \ldots, \mu_{a^*-1})$. Vector μ summarizes mortality in period t, while distribution x summarizes alive deprivation in period t as well as mortality before period t. The set of mortality vectors is defined as:

$$M = \left\{ \mu \in [0,1]^{a^*} \middle| \mu_{a^*-1} = 1 \right\}.$$

We consider pairs (x, μ) for which the distribution x is a priori unrelated to vector μ. We assume that the age-specific mortality rates μ_a must be feasible given the number of alive individuals $n_a(x)$. Given that distributions have finite numbers of individuals, mortality rates cannot take irrational values, i.e. $\mu_a \in [0,1] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. The set of pairs considered is given by:

$$O = \left\{ (x,\mu) \in X \times M \middle| \text{for all } a \in \{0,\ldots,a^*\} \text{ we have } \mu_a = \frac{c_a}{n_a(x)} \text{ for some } c_a \in \mathbb{N} \right\}.$$

Letting $d_a(x)$ be the number of *dead* individuals born a years before t in distribution x, the total number of individuals born a years before t is then equal to $n_a(x) + d_a(x)$. Formally, the **pair** (x, μ) is **stationary** if, for some $n^* \in \mathbb{N}$ and all $a \in \{0, \ldots, a^*\}$, we have:

- $n_a(x) + d_a(x) = n^* \in \mathbb{N}$ (constant natality),
- $n_{a+1}(x) = n_a(x) * (1 - \mu_a)$ (identical population pyramid in $t+1$).

In a stationary pair, the population pyramid is such that the size of each cohort can be obtained by applying to the preceding cohort the current mortality rate. The pair associated to a stationary society (as defined in the main text) is stationary. An index is a function $P : O \times \mathbb{N} \to \mathbb{R}_+$. We simplify the notation $P(x, \mu, \hat{a})$ to $P(x, \mu)$ as a fixed value for $\hat{a}$ is assumed.

We now introduce the properties characterizing $ED_{\theta\hat{a}}$. $ID_{\theta\hat{a}}$ Equivalence requires that, as the current mortality (in period t) is the same as the mortality prevailing in the previous periods in stationary societies, any index defined on current mortality rates is equivalent to $ID_{\theta\hat{a}}$ in the case of a stationary pair:[27]

Deprivation axiom 1 ($ID_{\theta\hat{a}}$ Equivalence). *There exists some $\theta \in (0,1]$ and $\hat{a} \geq \underline{\hat{a}}$ such that for all $(x,\mu) \in O$ that are stationary we have $P(x,\mu) = ID_{\theta\hat{a}}(x)$.*

Independence of Dead requires that past mortality does not affect the index. More precisely, the presence of an additional dead individual in distribution x does not affect the index:

Deprivation axiom 2 (Independence of Dead). *For all $(x,\mu) \in O$ and $i \leq n(x)$, if $s_i = D$, then $P((x_i, x_{-i}), \mu) = P(x_{-i}, \mu)$.*

Independence of Birth Year requires that the index does not depend on the birth year of individuals, i.e. only their status matters. As Independence of Dead requires to disregard dead individuals, the only relevant information in x is whether an alive individual is poor or not.

Deprivation axiom 3 (Independence of Birth Year). *For all $(x,\mu) \in O$ and $i \leq n(x)$, if $s_i = s'_i$, then $P((x_i, x_{-i}), \mu) = P((x'_i, x_{-i}), \mu)$.*

Replication Invariance requires that, if a distribution is obtained by replicating another distribution several times, they both have the same deprivation when associated to the same mortality vector. By definition, a k-replication of distribution x is a distribution $x^k = (x, \ldots, x)$ for which x is repeated k times.

Deprivation axiom 4 (Replication Invariance). *For all $(x,\mu) \in O$ and $k \in \mathbb{N}$, $P(x^k, \mu) = P(x, \mu)$.*

Proposition 7 shows that these properties jointly characterize the $ED_{\theta\hat{a}}$ index.

Proposition 7 (Characterization of $ED_{\theta\hat{a}}$).
$P = ED_{\theta\hat{a}}$ if and only if P satisfies Independence of Dead, $ID_{\theta\hat{a}}$ Equivalence, Replication Invariance and Independence of Birth Year.

Proof. We first prove sufficiency. Proving that the $ED_{\theta\hat{a}}$ index satisfies Independence of Dead, Replication Invariance and Independence of Birth Year is straightforward and left to the reader. Finally, $ED_{\theta\hat{a}}$ index satisfies $ID_{\theta\hat{a}}$ Equivalence because $ED_{\theta\hat{a}}$ is equal to $GD_{\theta\hat{a}}$ in stationary populations (Proposition 5) and $GD_{\theta\hat{a}}$ satisfies $ID_{\theta\hat{a}}$ Equivalence (Proposition 2 in Baland et al. (2021)). (The pairs associated to stationary societies are stationary).

We now prove necessity. Take any pair $(x,\mu) \in O$. We construct another pair (x''', μ) that is stationary and such that $P(x''', \mu) = P(x, \mu)$ and $ED_{\theta\hat{a}}(x''', \mu) = ED_{\theta\hat{a}}(x, \mu)$. Given that (x''', μ) is stationary, we have by $ID_{\theta\hat{a}}$ Equivalence that $P(x''', \mu) = ID_{\theta\hat{a}}(x''', \mu)$ for some $\theta \in (0,1]$. As $ID_{\theta\hat{a}} = GD_{\theta\hat{a}} = ED_{\theta\hat{a}}$ for stationary pairs, we have $P(x''', \mu) = ED_{\theta\hat{a}}(x''', \mu)$ for some $\theta \in (0,1]$. If we can construct such pair (x''', μ), then $P(x, \mu) = ED_{\theta\hat{a}}(x, \mu)$ for some $\theta \in (0,1]$, the desired result.

[27] Recall that past mortality is recorded in distribution x while current mortality is recorded in vector μ. As vector μ is redundant in stationary pairs, in the sense that μ can be inferred from the population pyramid, the index can be computed on distribution x only. See Baland et al. (2021) for a complete motivation for this axiom.

We turn to the construction of the stationary pair (x''', μ), using two intermediary pairs (x', μ) and (x'', μ). One difficulty is to ensure that the mortality rates μ_a can be achieved in the stationary population given the number of alive individuals $n_a(x''')$, that is $\mu_a = \frac{c}{n_a(x''')}$ for some $c \in \mathbb{N}$.

We first construct a $n'-$replication of x that has sufficiently many alive individuals to meet this constraint. For any $a \in \{0, \ldots, a^* - 1\}$, take any naturals c_a and e_a such that $\mu_a = \frac{c_a}{e_a}$. Let $e = \prod_{j=0}^{a^*-1} e_j$, $n'_a = e \prod_{j=0}^{a-1}(1 - \frac{c_j}{e_j})$ and $n' = \sum_{j=0}^{a^*-1} n'_j$.[28] Let x' be a $n'-$replication of x. Letting $n^x = \sum_{j=0}^{a^*-1} n_j(x)$ be the number of alive individuals in distribution x, we have that x' has $n' * n^x$ alive individuals. We have $P(x', \mu) = P(x, \mu)$ by Replication Invariance.

We define x'' from x' by changing the birth years of *alive* individuals in such a way that (x'', μ) has a population pyramid that is stationary. Formally, we construct x'' with $n(x'') = n(x')$ such that

- dead individuals in x' are also dead in x'',
- alive individuals in x' are also alive in x'' and have the same status,
- the birth year of alive individuals are changed such that, for each $a \in \{0, \ldots, a^* - 1\}$, the number of a-years old individuals is $n' * n^x * \frac{\prod_{j=0}^{a-1}(1 - \frac{c_j}{e_j})}{\sum_{k=0}^{a^*-1} \prod_{j=0}^{k-1}(1 - \frac{c_j}{e_j})}$.[29]

One can check that (x'', μ) has a population pyramid corresponding to a stationary population and that each age group has a number of alive individuals in $\mathbb{N}$. We have $P(x'', \mu) = P(x', \mu)$ by Independence of Birth Year.

Define x''' from x'' by changing the number and birth years of *dead* individuals in such a way that (x''', μ) is stationary. To do so, place exactly $n_0(x'') - n_a(x'')$ dead individuals in each age group a. We have $P(x''', \mu) = P(x'', \mu)$ by Independence of Dead.

Together, we have that $P(x''', \mu) = P(x, \mu)$. Finally, by construction we have $H(x''') = H(x)$, which implies that $ED_{\theta\hat{a}}(x''', \mu) = ED_{\theta\hat{a}}(x, \mu)$. □

[28]These numbers imply that a constant natality of e newborns leads to a stationary population of n' alive individuals.

[29]Observe that $\sum_{k=0}^{a^*-1} \prod_{j=0}^{k-1}(1 - \frac{c_j}{e_j}) = LE$, implying that $e = \frac{n' * n^x}{\sum_{k=0}^{a^*-1} \prod_{j=0}^{k-1}(1 - \frac{c_j}{e_j})}$.